GENERALIZED ISMAIL'S ARGUMENT AND (f,g)-EXPANSION FORMULA

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Abstract. As further development of earlier works on the (f,g)-inversion, the present paper is devoted to the (f,g)-difference operator and the representation problem or an expansion formula of analytic functions. A recursive formula and the Leibniz formula for the (f,g)-difference operator of the product of two functions are established. The resulting expansion formula not only unifies the q-analogue of the Lagrange inversion formula of Gessel and Stanton (thus, a q-expansion formula of Liu) for q-series but also systematizes the "Ismail's argument". In the meantime, a rigorous analytic proof of the $(1-xy,x-y)$-expansion formula with respect to geometric series, along with a proof of the previously unknown fact that it is equivalent to a q-analogue of the Lagrange inversion formula due to Gessel and Stanton, is presented. As applications, new proofs of several well-known summation and transformation formulas are investigated.

1. Introduction

It is well known that the core of the classical Lagrange inversion formula (cf.[51, §7.32]) is to express the coefficients a_n in the expansion of

$$F(x)=\sum_{n=0}^{\infty}a_n\left(\frac{x}{\phi(x)}\right)^n \tag{1.1}$$

by

$$a_n=n!\frac{d^{n-1}}{dx^{n-1}}\left[\phi^n(x)\frac{dF(x)}{dx}\right]_{x=0}$$

provided that $F(x)$ and $\phi(x)$ are analytic around $x=0$, $\phi(0)\neq 0$, $\frac{d}{dx}$ denotes the usual derivative operator.

In the past years, various q-analogues (as generalizations) of the Lagrange inversion formula, as an active field of research with an increasing number of applications to q-series and the Rogers-Ramanujan identities, have been studied by numerous authors (cf.[2, 11, 23, 29, 30, 46, 48]). For a good survey about results and open problems on this topic, we would like to refer the reader to Stanton's paper [48] and only repeat here, for reference purposes later, three noteworthy results.

One q-analogue found by Carlitz in 1973 (cf.[12, Eq.(1.11)]), subsequently reproduced by Roman (cf.[42, p. 253, Eq. (8.4)]) viz q-umbral calculus, is that for any formal series $F(x)$, it holds that

$$F(x)=\sum_{k=0}^{\infty}\frac{x^k}{(q,x;q)_k}\left[\mathcal{D}_{q,x}^{k}\{F(x)(x;q)_{k-1}\}\right]_{x=0}, \tag{1.2}$$

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where $\mathcal{D}_{q,x}$ denotes the q-difference operator.

Ten years later, Gessel and Stanton [23] successfully, with insight that the essential character of the Lagrange inversion formula is equivalent to finding the inverse of an infinite lower triangular matrix $F = (B_{n,k})$ subject to $B_{n,k} = 0$ unless $n \geq k$, $B_{n,n} \neq 0$, that is, another unique matrix $G = (B^{-1}_{n,k})$ satisfying

$$\sum\nolimits_{n\geq i\geq k} B^{-1}_{n,i} B_{i,k} = \delta_{n,k},$$

where δ denotes the usual Kronecker delta (as always, such a pair of F and G is called a *matrix inversion*), discovered a few q-analogues of some matrix inversions. See [23, Theorems 3.7 and 3.15]. One result is that for any formal power series $F(x)$,

$$F(x) = \sum_{n\geq k\geq 0}^{\infty} a_k \frac{(Ap^k q^k;p)_{n-k}}{(q;q)_{n-k}} q^{-nk} x^n \tag{1.3}$$

if and only if

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} q^{\binom{n-k+1}{2}+nk} \frac{(1-Ap^k q^k)(Aq^n p^{n-1};p^{-1})_{n-k-1}}{(q;q)_{n-k}} F(q^k). \tag{1.4}$$

In 2002, Liu [34] established by using of Carlitz's q-analogue (1.2) and the special case $c \mapsto \infty$ of Rogers' ${}_6\phi_5$ summation formula [22, II.20], as well as the q-difference operator, the following q-expansion formula: given any formal series $F(x)$, it holds that

$$F(x) = \sum_{k=0}^{\infty} \frac{(1-aq^{2k})(aq/x;q)_k x^k}{(q;q)_k (x;q)_k} \left[\mathcal{D}^k_{q,x}\{F(x)(x;q)_{k-1}\}\right]_{x=aq}. \tag{1.5}$$

As of today, these expansion formulas have been proved to be very important to the theory of basic hypergeometric series (i.e., q-series), but they were established without considering convergence and are therefore only valid within the ring of formal power series. See the above references for more details. The investigation of their rigorous analytic proofs, which is not only unavoidable but also essential to basic hypergeometric series and special function, is one of the purposes of the present paper.

To achieve this goal, one important idea we will invoke, particular regarding elementary derivations of summation and transformation formulas of basic hypergeometric series, is Ismail's analytic proof of Ramanujan's ${}_1\varphi_1$ summation formula [25]. It was often referred to as the "Ismail's argument" (cf.[44]) since it is Ismail who was apparently the first to apply analytic continuation argument in the context of bilateral basic hypergeometric series. Later, Askey and Ismail used this method and Rogers' ${}_6\phi_5$ summation formula [4] to evaluate Bailey's ${}_6\varphi_6$ sum. In principle, the "Ismail's argument" can be summarized briefly as follows: if one wants to prove two analytic functions $F(x) = G(x)$, all that is necessary is to show that they agree infinitely often near a point that is an interior point of the set of analyticity. Unfortunately, application of this idea to q-series has not been investigated systematically. Even later, it was treated extensively by Gasper in [20], from which one can see how the analytic continuation of a given function affects existences of summation and transformation formulas of q-series, but this method has not yet been written in their remarkable book [22]. One reason for this is that, adopting Gasper as saying" $\cdots$ the succeeding higher order derivatives becomes more and more difficult to calculate for $|z| < 1$, and so one is forced to abandon this approach and to search for another way $\cdots$" . See [20] for more details.

The purposes of the present paper are: (1) to show that all above q-analogues or expansion formulas can be unified by, from a purely analytic viewpoint, the representation problem of

$F(x)$ in terms of a new kind of series, that is $(f,g)-$series (see Definition 3.2 below),

$$F(x) \sim \sum_{n=0}^{\infty} G(n) f(x_n, b_n) \frac{\prod_{i=0}^{n-1} g(b_i, x)}{\prod_{i=1}^{n} f(x_i, x)} \tag{1.6}$$

where the coefficient $G(n)$ denotes the n-th order (f,g)-difference of $F(x)$ (see Definition 2.1 below) expressed explicitly by (3.4), $f(x,y) \in Ker\mathcal{L}_3^{(g)}$ the set of functions over $\mathbb{C} \times \mathbb{C}$ in two variables x, y, $\mathbb{C}$ is the usual complex field, such that for all $a, b, c, x \in \mathbb{C}$

$$f(x,a)g(b,c) + f(x,b)g(c,a) + f(x,c)g(a,b) = 0, \tag{1.7}$$

where $g(x,y)$ is antisymmetric, i.e., $g(x,y) = -g(y,x)$. Later as we will see, they are special cases of the following expansion formula

$$F(x) = \sum_{n=0}^{\infty} G(n) f(x_n, b_n) \frac{\prod_{i=0}^{n-1} g(b_i, x)}{\prod_{i=1}^{n} f(x_i, x)} \tag{1.8}$$

under the assumption that the series in the r.h.s. of (1.6) converges to $F(x)$. In a certain sense, it provides a *discrete* analogue of the Lagrange inversion formula by replacing $(x/\phi(x))^n$ in (1.1) with $\prod_{i=0}^{n-1} g(b_i, x) / \prod_{i=1}^{n} f(x_i, x)$. (2) to set up an existence theorem of (1.8) when $F(x)$, $\{b_i\}$, and $\{x_i\}$ are subject to suitable convergence conditions. We believe it can serve as a standard model for the "Ismail's argument". We call it the generalized "Ismail's argument". (3) to apply the generalized "Ismail's argument" to basic hypergeometric series in order to derive new or review known summation formulas.

Our argument is based on a recent discovery that: as our previously paper [36] shows, Identity (1.7) implies the following matrix inversion, and vice versa. It is called the (f,g)-inversion formally in [38] by the author.

Theorem 1.1. *Let $F = (B_{n,k})_{n,k\in\mathbb{Z}}$ and $G = (B^{-1}_{n,k})_{n,k\in\mathbb{Z}}$ be two matrices with entries given by*

$$B_{n,k} = \frac{\prod_{i=k}^{n-1} f(x_i, b_k)}{\prod_{i=k+1}^{n} g(b_i, b_k)} \quad \text{and} \tag{1.9}$$

$$B^{-1}_{n,k} = \frac{f(x_k, b_k)}{f(x_n, b_n)} \frac{\prod_{i=k+1}^{n} f(x_i, b_n)}{\prod_{i=k}^{n-1} g(b_i, b_n)}, \quad \text{respectively,} \tag{1.10}$$

where $\mathbb{Z}$ denotes the set of integers, $\{x_i\}$ and $\{b_i\}$ are arbitrary sequences such that none of the denominators in the right hand sides of (1.9) and (1.10) vanish. Then $F = (B_{n,k})_{n,k\in\mathbb{Z}}$ and $G = (B^{-1}_{n,k})_{n,k\in\mathbb{Z}}$ is a matrix inversion if and only if for all $a, b, c, x \in \mathbb{C}$, (1.7) holds.

The reader may consult [36, 38] for more detailed expositions.

The present paper, as further study of the (f,g)-inversion, is organized as follows. In Section 2, the n-th order (f,g)-difference operator is introduced and some basic propositions like the *Leibniz formula* are examined. Section 3 is devoted to the generalized "Ismail's argument" and some (f,g)-expansion formulas. They are all special solutions of the representation problem of analytic functions in terms of (f,g)-series. A rigorous analytic proof of the $(1-xy, x-y)$-expansion formula with respect to two sequences of geometric series as well as a proof of the previously unknown fact that it is equivalent to the q-analogue (1.3)/(1.4) of the Lagrange inversion formula due to Gessel and Stanton are given in Sections 4 and 7, respectively. Their applications to basic hypergeometric series are investigated in details in Section 5. In Section 6, a few concluding remarks on some problems related with the (f,g)-expansion formulas are given.

Notations and conventions. Throughout this paper we adopt the standard notation and terminology for basic hypergeometric series of Gasper and Rahman's book [22], which is also a main reference for any related results concerning basic hypergeometric series. For instance,

given a (fixed) complex number q with $|q|<1$, a complex a and a natural number n, define the q-shifted factorials $(a;q)_n$ and $(a;q)_\infty$ as

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),\quad (a;q)_n=(a;q)_\infty/(aq^n;q)_\infty$$

with the following compact multi-parameter notation

$$(a_1,a_2,\cdots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$$

and the q-binomial coefficient $\begin{bmatrix}n\\k\end{bmatrix}_q=(q;q)_n/((q;q)_k(q;q)_{n-k})$. The *basic hypergeometric series* with the base q is defined by

$${}_r\phi_s\left[\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix};q,z\right]=\sum_{n=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n}\left[(-1)^nq^{n(n-1)/2}\right]^{1+s-r}z^n,$$

while the *bilateral basic hypergeometric series* is defined as

$${}_r\varphi_s\left[\begin{matrix}a_1,\ldots,a_r\\b_1,\ldots,b_s\end{matrix};q,z\right]=\sum_{n=-\infty}^{\infty}\frac{(a_1,\cdots,a_r;q)_n}{(b_1,\cdots,b_s;q)_n}\left[(-1)^nq^{n(n-1)/2}\right]^{s-r}z^n.$$

In the setting of bilateral series, we employ the convention of defining (a product form of Eq.(3.6.12) in [22]) over $\mathbb{Z}$

$$\prod_{j=k}^{m}A_j=\begin{cases}A_kA_{k+1}\cdots A_m, & m\geq k;\\ 1, & m=k-1;\\ (A_{m+1}A_{m+2}\cdots A_{k-1})^{-1}, & m\leq k-2.\end{cases}$$

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2. The n-th order (f,g)-difference operator

Before we turn to our main results, we need the definition of (f,g)-difference operator which will play the same role in our expansion formulas as the divided difference in the classical Newton's interpolation formula. In what follows, we use $\mathbb{C}(x)$ to denote the linear space of all functions over $\mathbb{C}$ of a variable x.

Definition 2.1. *Let $f(x,y)\in Ker\mathcal{L}_3^{(g)}$, $\{x_i\}$ and $\{b_i\}$ be arbitrary sequences such that none of the denominators in (2.1) vanish. Then the mapping*

$$\mathbb{D}_{(f,g)}^{(n)}\left[\begin{matrix}b_0,b_1,\ldots,b_n\\x_1,\ldots,x_{n-1}\end{matrix}\right]\{\bullet\}:\mathbb{C}(x)\longrightarrow\mathbb{C},$$

such that

$$\mathbb{D}_{(f,g)}^{(n)}\left[\begin{matrix}b_0,b_1,\ldots,b_n\\x_1,\ldots,x_{n-1}\end{matrix}\right]\{F(x)\}=\sum_{k=0}^{n}F(b_k)\frac{\prod_{i=1}^{n-1}f(x_i,b_k)}{\prod_{i=0,i\neq k}^{n}g(b_i,b_k)}\tag{2.1}$$

is said to be the n-th order (f,g)-difference operator with respect to $n+1$ pairwise distinct nodes $b_0,b_1,b_2,\cdots,b_n$ and $n-1$ parameters $x_1,x_2,\cdots,x_{n-1}$.

In what follows, we often abbreviate $\mathbb{D}_{(f,g)}^{(n)}\left[\begin{matrix}b_0,b_1,\ldots,b_n\\x_1,\ldots,x_{n-1}\end{matrix}\right]\{\bullet\}$ by $\mathbb{D}_{(f,g)}^{(n)}$ if it is clear in the context. When $f=1$ or $n=1$, the notation $\mathbb{D}_{(g)}^{(n)}[b_0,b_1,b_2,\cdots,b_n]$ is employed in which $\{x_i\}$

are bypassed, since in either of these two cases, the sum does certainly not depend on $\{x_i\}$. In particular, for $n = 0$, by the convention,

$$\mathbb{D}^{(n)}_{(f,g)} \begin{bmatrix} b_0 \\ x_{-1} \end{bmatrix} \{F(x)\} = \frac{F(b_0)}{f(x_0, b_0)}.$$

The following example displays that $\mathbb{D}^{(n)}_{(f,g)}$ is a generalization of the divided difference and the q-difference operator in numerical analysis and special function.

Example 2.1. *Let $f(x, y) = 1, g(x, y) = x - y, b_i = x_{i+1}, i = 0, 1, 2, \cdots$. Then*

$$\mathbb{D}^{(n)}_{(g)}[x_1, x_2, \cdots, x_{n+1}]\{F(x)\} = (-1)^n F[x_1, x_2, \cdots, x_{n+1}], \tag{2.2}$$

where the classical divided difference $F[x_1, x_2, \cdots, x_{n+1}]$ is recursively defined by

$$\begin{aligned} &F[x_1] = F(x_1);\\ &F[x_1, x_2] = \frac{F(x_1) - F(x_2)}{x_1 - x_2};\\ &F[x_1, x_2, x_3] = \frac{F[x_1, x_2] - F[x_2, x_3]}{x_1 - x_3};\\ &\cdots;\\ &F[x_1, x_2, \cdots, x_n] = \frac{F[x_1, x_2, \cdots, x_{n-1}] - F[x_2, x_3, \cdots, x_n]}{x_1 - x_n}. \end{aligned}$$

See [9, p.123] *for more details.*

As might be expected from this definition by specializing $b_i = x+h(i-1)$, $\mathbb{D}^{(n)}_{(g)}$ can be expressed in a simpler form

$$\mathbb{D}^{(n)}_{(g)}[x, x + h, x + 2h, \cdots, x + nh]\{F(x)\} = \frac{1}{n!h^n} \nabla^n_h F(x),$$

where ∇_h is the usual backward difference operator defined as $\nabla_h\{F(x)\} = F(x) - F(x + h)$. In the meantime, assume that $F(x)$ is n-times differentiable at x. Then it is easily found that

$$\lim_{h\to 0} \mathbb{D}^{(n)}_{(g)}[x, x + h, x + 2h, \cdots, x + nh]\{F(x)\} = \frac{1}{n!} \frac{d^n}{dx^n} F(x). \tag{2.3}$$

Another interesting case is that with the specification $b_i = xq^i, i = 0, 1, 2, \cdots$, we have

$$\mathbb{D}^{(1)}_{(g)}[x, xq]\{F(x)\} = \frac{1}{(q - 1)} \mathcal{D}_{q,x} F(x),$$

where $\mathcal{D}_{q,x}$ denotes the q-difference operator (cf.[28]) *appeared previously in (1.2)*

$$\mathcal{D}_{q,x} F(x) = \frac{F(x) - F(qx)}{x}. \tag{2.4}$$

Therefore, from the above observations, we conclude that the n-th order (f, g)-difference operator $\mathbb{D}^{(n)}_{(f,g)}$ is particularly useful and deserves further investigation. As a consequence, we find that $\mathbb{D}^{(n)}_{(f,g)}$ satisfies the following recursive formula previously well known as a basic property of the classical divided difference.

Theorem 2.1. *Let $f(x, y) \in Ker\mathcal{L}^{(g)}_3$, and $\mathbb{D}^{(n)}_{(f,g)}$ be defined as above. Then for any integer $n \geq 0$,*

$$\begin{aligned} \mathbb{D}^{(n+1)}_{(f,g)} \begin{bmatrix} b_0, b_1, \ldots, b_{n+1} \\ x_1, \ldots, x_n \end{bmatrix} \{F(x)\} &= \frac{f(x_n, b_0)}{g(b_{n+1}, b_0)} \mathbb{D}^{(n)}_{(f,g)} \begin{bmatrix} b_0, b_1, \ldots, b_n \\ x_1, \ldots, x_{n-1} \end{bmatrix} \{F(x)\} \\ &+ \frac{f(x_n, b_{n+1})}{g(b_0, b_{n+1})} \mathbb{D}^{(n)}_{(f,g)} \begin{bmatrix} b_1, b_2, \ldots, b_{n+1} \\ x_1, \ldots, x_{n-1} \end{bmatrix} \{F(x)\}. \end{aligned} \tag{2.5}$$

In particular, if $f(x,y)=1\in Ker\mathcal{L}_3^{(g)}$, then

$$\begin{aligned}(2.6)\qquad &\mathbb{D}_{(g)}^{(n+1)}[b_0,b_1,b_2,\cdots,b_n,b_{n+1}]\{F(x)\}\\ =\ &\frac{\mathbb{D}_{(g)}^{(n)}[b_0,b_1,b_2,\cdots,b_n]\{F(x)\}-\mathbb{D}_{(g)}^{(n)}[b_1,b_2,b_3,\cdots,b_{n+1}]\{F(x)\}}{g(b_{n+1},b_0)}.\end{aligned}$$

Proof. By Definition 2.1, a straightforward calculation leads us to

$$\begin{aligned}&\mathbb{D}_{(f,g)}^{(n+1)}\begin{bmatrix}b_0,b_1,\ldots,b_{n+1}\\ x_1,\ldots,x_n\end{bmatrix}\{F(x)\}-\frac{f(x_n,b_0)}{g(b_{n+1},b_0)}\mathbb{D}_{(f,g)}^{(n)}\begin{bmatrix}b_0,b_1,\ldots,b_n\\ x_1,\ldots,x_{n-1}\end{bmatrix}\{F(x)\}\\ =\ &\sum_{k=0}^{n+1}F(b_k)\frac{\prod_{i=1}^{n}f(x_i,b_k)}{\prod_{i=0,i\neq k}^{n+1}g(b_i,b_k)}-\frac{f(x_n,b_0)}{g(b_{n+1},b_0)}\sum_{k=0}^{n}F(b_k)\frac{\prod_{i=1}^{n-1}f(x_i,b_k)}{\prod_{i=0,i\neq k}^{n}g(b_i,b_k)}\\ =\ &F(b_{n+1})\frac{\prod_{i=1}^{n}f(x_i,b_{n+1})}{\prod_{i=0}^{n}g(b_i,b_{n+1})}+\sum_{k=0}^{n}\left\{\frac{f(x_n,b_k)}{g(b_{n+1},b_k)}-\frac{f(x_n,b_0)}{g(b_{n+1},b_0)}\right\}F(b_k)\frac{\prod_{i=1}^{n-1}f(x_i,b_k)}{\prod_{i=0,i\neq k}^{n}g(b_i,b_k)}.\end{aligned}$$

Note that the term within the curly braces can be simplified by (1.7), which arises from $f(x,y)\in Ker\mathcal{L}_3^{(g)}$. The result is

$$\begin{aligned}&\sum_{k=0}^{n}\left\{\frac{f(x_n,b_k)}{g(b_{n+1},b_k)}-\frac{f(x_n,b_0)}{g(b_{n+1},b_0)}\right\}F(b_k)\frac{\prod_{i=1}^{n-1}f(x_i,b_k)}{\prod_{i=0,i\neq k}^{n}g(b_i,b_k)}\\ =\ &-\frac{f(x_n,b_{n+1})}{g(b_{n+1},b_0)}\sum_{k=1}^{n}F(b_k)\frac{\prod_{i=1}^{n-1}f(x_i,b_k)}{\prod_{i=1,i\neq k}^{n+1}g(b_i,b_k)}.\end{aligned}$$

Simplify the preceding identity by this fact to obtain

$$\begin{aligned}&-\frac{f(x_n,b_{n+1})}{g(b_{n+1},b_0)}\sum_{k=1}^{n}F(b_k)\frac{\prod_{i=1}^{n-1}f(x_i,b_k)}{\prod_{i=1,i\neq k}^{n+1}g(b_i,b_k)}+F(b_{n+1})\frac{\prod_{i=1}^{n}f(x_i,b_{n+1})}{\prod_{i=0}^{n}g(b_i,b_{n+1})}\\ =\ &\frac{f(x_n,b_{n+1})}{g(b_0,b_{n+1})}\sum_{k=1}^{n+1}F(b_k)\frac{\prod_{i=1}^{n-1}f(x_i,b_k)}{\prod_{i=1,i\neq k}^{n+1}g(b_i,b_k)}.\end{aligned}$$

By Definition 2.1, the sum in the r.h.s. of this identity is none other than $\mathbb{D}_{(f,g)}^{(n)}\begin{bmatrix}b_1,b_2,\ldots,b_{n+1}\\ x_1,\ldots,x_{n-1}\end{bmatrix}\{F(x)\}$. This gives

$$\begin{aligned}\mathbb{D}_{(f,g)}^{(n+1)}\begin{bmatrix}b_0,b_1,\ldots,b_{n+1}\\ x_1,\ldots,x_n\end{bmatrix}\{F(x)\}\ &-\ \frac{f(x_n,b_0)}{g(b_{n+1},b_0)}\mathbb{D}_{(f,g)}^{(n)}\begin{bmatrix}b_0,b_1,\ldots,b_n\\ x_1,\ldots,x_{n-1}\end{bmatrix}\{F(x)\}\\ &=\ \frac{f(x_n,b_{n+1})}{g(b_0,b_{n+1})}\mathbb{D}_{(f,g)}^{(n)}\begin{bmatrix}b_1,b_2,\ldots,b_{n+1}\\ x_1,\ldots,x_{n-1}\end{bmatrix}\{F(x)\}.\end{aligned}$$

So the theorem is proved. ∎

A corollary is obtained immediately from the argument of this theorem.

Corollary 2.1. *Preserve the assumption as above. Then for $n\geq m$,*

$$(2.7)\qquad \mathbb{D}_{(f,g)}^{(n)}\begin{bmatrix}b_0,b_1,\ldots,b_n\\ x_1,\ldots,x_{n-1}\end{bmatrix}\left\{\prod_{i=0}^{m-1}g(b_i,x)/\prod_{i=1}^{m}f(x_i,x)\right\}=\frac{1}{f(x_m,b_m)}\delta_{n,m};$$

In particular, for $n\geq 1$,

$$(2.8)\qquad \mathbb{D}_{(f,g)}^{(n)}\begin{bmatrix}b_0,b_1,\ldots,b_n\\ x_1,\ldots,x_{n-1}\end{bmatrix}\{1\}=0.$$

Proof. It can be proved by Theorem 2.1 and induction on n. Omitted here. ∎

Concerning the nth order (f,g)-difference of the product of two functions, the next theorem states that $\mathbb{D}^{(n)}_{(f,g)}$ also satisfies the *Leibniz formula*:

Theorem 2.2. *Let $f(x,y) \in Ker\mathcal{L}^{(g)}_3$, and $\mathbb{D}^{(n)}_{(f,g)}$ be defined as above. Then*

$$\mathbb{D}^{(n)}_{(f,g)}\begin{bmatrix} b_0, b_1, \ldots, b_n \\ x_1, \ldots, x_{n-1}\end{bmatrix}\{F(x)H(x)\} \tag{2.9}$$
$$= \sum_{k=0}^{n} f(x_k,b_k)\,\mathbb{D}^{(k)}_{(f,g)}\begin{bmatrix} b_0, b_1, \ldots, b_k \\ x_1, \ldots, x_{k-1}\end{bmatrix}\{H(x)\}\,\mathbb{D}^{(n-k)}_{(f,g)}\begin{bmatrix} b_k, b_{k+1}, \ldots, b_n \\ x_{k+1}, \ldots, x_{n-1}\end{bmatrix}\{F(x)\}.$$

Proof. Observe that the r.h.s. of (2.9) can be reformulated as

$$\sum_{0\le i\le j\le n} \lambda_{i,j} H(b_i)F(b_j), \tag{2.10}$$

where the coefficients $\lambda_{i,j}$ are given by

$$\begin{aligned}
\lambda_{i,j} &= \sum_{k=i}^{j} f(x_k,b_k)\frac{\prod_{l=1}^{k-1} f(x_l,b_i)}{\prod_{i_1=0,i_1\neq i}^{k} g(b_{i_1},b_i)}\frac{\prod_{l=k+1}^{n-1} f(x_l,b_j)}{\prod_{i_2=k,i_2\neq j}^{n} g(b_{i_2},b_j)}\\
&= \frac{\prod_{l=1}^{i-1} f(x_l,b_i)}{\prod_{i_1=0}^{i-1} g(b_{i_1},b_i)}\frac{\prod_{l=j}^{n-1} f(x_l,b_j)}{\prod_{i_2=j+1}^{n} g(b_{i_2},b_j)}\\
&\times \sum_{k=i}^{j}\frac{\prod_{l=i}^{k-1} f(x_l,b_i)}{\prod_{i_1=i+1}^{k} g(b_{i_1},b_i)}\left\{f(x_k,b_k)\frac{\prod_{l=k+1}^{j-1} f(x_l,b_j)}{\prod_{i_2=k}^{j-1} g(b_{i_2},b_j)}\right\}.
\end{aligned}$$

The last sum turns out to be $\delta_{j,i}$, as the (j,i)-th entry of the product of a pair of matrices $(B^{-1}_{j,k})$ and $(B_{k,i})$ given in Theorem 1.1. It reduces the preceding sum to

$$\sum_{0\le i\le j\le n} \lambda_{i,j} H(b_i)F(b_j) = \sum_{i=0}^{n} H(b_i)F(b_i)\frac{\prod_{l=1}^{n-1} f(x_l,b_i)}{\prod_{i_1=0,i_1\neq i}^{n} g(b_{i_1},b_i)}, \tag{2.11}$$

which is just, by Definition 2.1, the desired result. ∎

Explicit expressions of $\mathbb{D}^{(n)}_{(f,g)}$ for $n = 1, 2$ will now be given as examples to justify this formula.

Example 2.2. *Let $\mathbb{D}^{(n)}_{(f,g)}$ be given by (2.1). Then*

$$\mathbb{D}^{(1)}_{(g)}[b_0,b_1]\{F(x)H(x)\} = H(b_0)\,\mathbb{D}^{(1)}_{(g)}[b_0,b_1]\{F(x)\} + F(b_1)\,\mathbb{D}^{(1)}_{(g)}[b_0,b_1]\{H(x)\}; \tag{2.12}$$

$$\mathbb{D}^{(2)}_{(f,g)}\begin{bmatrix} b_0, b_1, b_2 \\ x_1\end{bmatrix}\{F(x)H(x)\} = f(x_1,b_1)\,\mathbb{D}^{(1)}_{(f,g)}[b_0,b_1]\{H(x)\}\,\mathbb{D}^{(1)}_{(f,g)}[b_1,b_2]\{F(x)\}$$
$$+H(b_0)\,\mathbb{D}^{(2)}_{(f,g)}\begin{bmatrix} b_0, b_1, b_2 \\ x_1\end{bmatrix}\{F(x)\} + F(b_2)\,\mathbb{D}^{(2)}_{(f,g)}\begin{bmatrix} b_0, b_1, b_2 \\ x_1\end{bmatrix}\{H(x)\}. \tag{2.13}$$

We end this section with two connections between $\mathbb{D}^{(n)}_{(x-y)}$ and $\mathcal{D}_{q,x}$, as well as new but short proofs of a few basic propositions of $\mathcal{D}_{q,x}$. Different from all known arguments is that we only utilize the definition of the (f,g)-difference operator.

Proposition 2.1. *Let $\mathbb{D}^{(n)}_{(g)}$ be given by (2.1). Then the following hold for $n \geq 1, m \geq 0$,*

$$\mathbb{D}^{(n)}_{(x-y)}[x, xq, xq^2, \cdots, xq^n]\{F(x)\} = \frac{(-1)^n}{(q;q)_n}\mathcal{D}^n_{q,x}\{F(x)\}; \tag{2.14}$$

$$\mathbb{D}^{(n)}_{(x-y)}[xq^m, xq^{m+1}, xq^{m+2}, \cdots, xq^{n+m}]\{F(x)\} = \frac{(-1)^n q^{-nm}}{(q;q)_n}\mathcal{D}^n_{q,x}\{F(xq^m)\}. \tag{2.15}$$

Proof. We proceed to show (2.14) in question by induction on n. At first, by the definition (see (2.4)), it is easy to check that

$$\begin{aligned}
\mathcal{D}_{q,x}F(x) &= \frac{F(x)-F(qx)}{x} = (q-1)\frac{F(x)-F(qx)}{x(q-1)} = (q-1)\,\mathbb{D}^{(1)}_{(x-y)}[x, xq]\{F(x)\};\\
\mathcal{D}^2_{q,x}\{F(x)\} &= \mathcal{D}_{q,x}\{\mathcal{D}_{q,x}F(x)\} = \frac{qF(x)-(1+q)F(qx)+F(q^2x)}{qx^2}\\
&= (q;q)_2\,\mathbb{D}^{(2)}_{(x-y)}[x, xq, xq^2]\{F(x)\}.
\end{aligned}$$

Hence, the result holds for $n = 1, 2$. Now, assume that the result in question is valid for $n = k$. Thus, we have

$$\mathbb{D}^{(k)}_{(x-y)}[x, xq, xq^2, \cdots, xq^k]\{F(x)\} = \frac{(-1)^k}{(q;q)_k}\mathcal{D}^k_{q,x}\{F(x)\}.$$

Next, consider $n = k+1$. For this case, from (2.5), it follows that

$$\begin{aligned}
\mathbb{D}^{(k+1)}_{(x-y)}[x, xq, xq^2, \cdots, xq^{k+1}]\{F(x)\} &= \frac{1}{xq^{k+1}-x}\,\mathbb{D}^{(k)}_{(x-y)}[x, xq, xq^2, \cdots, xq^k]\{F(x)\}\\
&\quad+\frac{1}{x-xq^{k+1}}\,\mathbb{D}^{(k)}_{(x-y)}[xq, xq^2, \cdots, xq^{k+1}]\{F(x)\}.
\end{aligned}$$

Observe that

$$\mathbb{D}^{(k)}_{(x-y)}[xq, xq^2, \cdots, xq^{k+1}]\{F(x)\} = \mathbb{D}^{(k)}_{(x-y)}[x, xq, xq^2, \cdots, xq^k]\{F(x)\}|_{x\mapsto qx},$$

where $x \mapsto qx$ denotes the replacement of x with xq. Insert it into the preceding identity to arrive at

$$\begin{aligned}
&\mathbb{D}^{(k+1)}_{(x-y)}[x, xq, xq^2, \cdots, xq^{k+1}]\{F(x)\}\\
&= \frac{1}{q^{k+1}-1}\frac{\mathbb{D}^{(k)}_{(x-y)}[x, xq, xq^2, \cdots, xq^k]\{F(x)\} - \mathbb{D}^{(k)}_{(x-y)}[xq, xq^2, \cdots, xq^{k+1}]\{F(xq)\}}{x}\\
&= \frac{1}{q^{k+1}-1}\mathcal{D}_{q,x}\left\{\mathbb{D}^{(k)}_{(x-y)}[x, xq, xq^2, \cdots, xq^k]\{F(x)\}\right\} = \frac{(-1)^{k+1}}{(q;q)_{k+1}}\mathcal{D}_{q,x}\left\{\mathcal{D}^k_{q,x}\{F(x)\}\right\}\\
&= \frac{(-1)^{k+1}}{(q;q)_{k+1}}\mathcal{D}^{k+1}_{q,x}\{F(x)\}.
\end{aligned}$$

Note that the last identity comes from the induction hypothesis. Thus, the desired result holds also for $n = k+1$. Summing up, this gives the complete proof of (2.14).

The correctness of (2.15) is proved by noting that

$$\begin{aligned}
&\mathbb{D}^{(n)}_{(x-y)}[xq^m, xq^{m+1}, xq^{m+2}, \cdots, xq^{n+m}]\{F(t)\}\\
&= q^{-nm}\,\mathbb{D}^{(n)}_{(x-y)}[x, xq, xq^2, \cdots, xq^n]\{F(tq^m)\}
\end{aligned}$$

and applying (2.14) to the r.h.s. of this identity. ∎

Further, the (f, g)-difference operator allows to evaluate $\mathcal{D}^n_{q,x}$ explicitly. As Koornwinder pointed out in his unpublished [32], it was first obtained in 1921 by Ryde [43].

Proposition 2.2. *Let $\mathcal{D}_{q,x}$ be the usual q-difference operator. Then for $n \geq 1$,*

$$\mathcal{D}_{q,x}^{n}\left\{F(x)\right\} = \frac{1}{x^n}\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-nk}\begin{bmatrix} n \\ k \end{bmatrix}_q F(xq^k). \tag{2.16}$$

Proof. Use Proposition 2.1 to calculate directly

$$\mathcal{D}_{q,x}^{n}\left\{F(x)\right\} = (-1)^n (q;q)_n \mathbb{D}_{(g)}^{(n)}[x, xq, xq^2, \cdots, xq^n]\{F(x)\}$$
$$= (-1)^n(q;q)_n \sum_{k=0}^{n}\frac{F(xq^k)}{\prod_{i=0,i\neq k}^{n}(xq^i - xq^k)} = \frac{1}{x^n}\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-nk}\begin{bmatrix} n \\ k \end{bmatrix}_q F(xq^k).$$

∎

By (2.16) and the following finite form of the q-binomial theorem [3, p.490, Corollary 10.2.2, Eq.(c)]

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x;q)_n, \tag{2.17}$$

it is immediately seen that

Proposition 2.3. *Let $F(x)$ be a polynomial of degree less than m in x. Then for any integer $n \geq m+1$,*

$$\mathcal{D}_{q,x}^{n}\left\{F(x)\right\} = 0. \tag{2.18}$$

Proof. Assume that $F(x) = \sum_{i=0}^{m} a_i x^i$. Then it is easily found that

$$\begin{aligned}\mathcal{D}_{q,x}^{n}\left\{F(x)\right\} &= \sum_{i=0}^{m} a_i \mathcal{D}_{q,x}^{n}\left\{x^i\right\} = \sum_{i=0}^{m} a_i x^{i-n}\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q q^{(1+i-n)k} \\ &= \sum_{i=0}^{m} a_i x^{i-n}(q^{(1+i-n)};q)_n = 0\end{aligned}$$

because that for $n \geq m+1 \geq i+1 \geq 1$, $(q^{(1+i-n)};q)_n = 0$. ∎

Taken together, (2.9) and (2.16) leads to

Proposition 2.4. (See [42, p.233]).

$$\mathcal{D}_{q,x}^{n}\left\{F(x)H(x)\right\} = \sum_{k=0}^{n} q^{(k-n)k}\begin{bmatrix} n \\ k \end{bmatrix}_q \mathcal{D}_{q,x}^{k}\left\{F(x)\right\}\mathcal{D}_{q,x}^{n-k}\left\{H(q^k x)\right\}. \tag{2.19}$$

3. The generalized Ismail's argument and (f,g)-expansion formulas

Now recall that as a basic application of matrix inversion, if the two lower-triangular matrices $(B_{n,k})$ and $(B_{n,k}^{-1})$ are inverses of each other, then for any two sequences $\{X_i\}$ and $\{Y_i\}$,

$$\sum_{k=0}^{n} B_{n,k}X_k = Y_n \quad \text{if and only if} \quad \sum_{k=0}^{n} B_{n,k}^{-1}Y_k = X_n. \tag{3.1}$$

Simple as it seems, many facts display that (3.1) provides a standard and powerful technique for deriving new summation formulas from known ones. The reader is referred to [8, 11, 23, 24, 29, 30, 31, 39, 45, 51] for further details.

In the sequel, once the above standard technique being applied to Theorem 1.1, it is not hard to set up the following special result on which our forthcoming discussions rely.

Lemma 3.1. *Let $f(x,y)$ and $g(x,y)$, $\{x_i\}$ and $\{b_i\}$ be given as in Theorem 1.1. Then the system of linear relations for any two sequences $\{X_i\}$ and $\{Y_i\}$*

$$X_n = \sum_{k=0}^{n} Y_k f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, b_n)}{\prod_{i=1}^{k} f(x_i, b_n)} \tag{3.2}$$

is equivalent to the system

$$Y_n = \sum_{k=0}^{n} X_k \frac{\prod_{i=1}^{n-1} f(x_i, b_k)}{\prod_{i=0, i\neq k}^{n} g(b_i, b_k)}. \tag{3.3}$$

Proof. Assume that

$$\sum_{k=0}^{n} X_k \frac{\prod_{i=1}^{n-1} f(x_i, b_k)}{\prod_{i=0, i\neq k}^{n} g(b_i, b_k)} = Y_n,$$

or, equivalently

$$\sum_{k=0}^{n} \frac{\prod_{i=k}^{n-1} f(x_i, b_k)}{\prod_{i=k+1}^{n} g(b_i, b_k)} \left\{ \frac{\prod_{i=1}^{k-1} f(x_i, b_k)}{\prod_{i=0}^{k-1} g(b_i, b_k)} X_k \right\} = Y_n.$$

Then by Theorem 1.1, i.e., the (f, g)-inversion, we have that

$$\sum_{k=0}^{n} Y_k f(x_k, b_k) \frac{\prod_{i=k+1}^{n-1} f(x_i, b_n)}{\prod_{i=k}^{n-1} g(b_i, b_n)} = \frac{\prod_{i=1}^{n-1} f(x_i, b_n)}{\prod_{i=0}^{n-1} g(b_i, b_n)} X_n.$$

Divide both sides of the relation by $\prod_{i=1}^{n-1} f(x_i, b_n) / \prod_{i=0}^{n-1} g(b_i, b_n)$ and simplify the resulted. It gives

$$X_n = \sum_{k=0}^{n} Y_k f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, b_n)}{\prod_{i=1}^{k} f(x_i, b_n)}.$$

Vice versa. ∎

If we set $X_n = F(x)|_{x=b_n}$, $F(x)$ is a known analytic function in certain region, as one of important special case of Lemma 3.1, then

$$Y_n = \mathbb{D}_{(f,g)}^{(n)} \begin{bmatrix} b_0, b_1, \ldots, b_n \\ x_1, \ldots, x_{n-1} \end{bmatrix} \{F(x)\}.$$

Related by such a pair of sequences, Identity (3.2) deserves a separate definition, which is in analogy to the Fourier series expansions of analytic functions. For this, we use Ω to denote an open subset of the complex plane, and $\mathcal{H}(\Omega)$ the space of analytic functions over Ω.

Definition 3.2. *With the assumption as above. Let $F(x) \in \mathcal{H}(\Omega)$. The following series*

$$\sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)} \tag{3.4}$$

is called the (f, g)-series generated by $F(x)$ with respect to two sequences $\{b_i\}$ and $\{x_i\}$ over Ω. We denote it by

$$F(x) \sim \sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)} \tag{3.5}$$

Very similar to the situation in the theory of Fourier series, we come up against two questions:

The convergence problem: Does the series (3.4) converge at some point $x \in \Omega$?
The representation problem: If (3.4) does converge at $x \in \Omega$, is its sum $F(x)$? more precisely,

$$F(x) = \sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)}. \tag{3.6}$$

In fact, when we deal with these two questions, we can not expect a simple, clear-cut answer without assuming further conditions. In the present paper, we are mainly concerned with the second problem with an effort to set up such expansion formulas. For this, we propose the following definition

Definition 3.3. *Let $F(x) \in \mathcal{H}(\Omega)$ and $f(x,y) \in Ker\mathcal{L}_3^{(g)}$. If there exist three sequences $\{x_i\},\{b_i\} \subseteq \Omega$, and $\{\chi(i)\}$, such that for any $x \in \Omega$,*

$$F(x) = \sum_{k=0}^{\infty} \chi(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)}. \tag{3.7}$$

then it is called the (f,g)-expansion formula of $F(x)$ with respect to two sequences $\{b_i\}$ and $\{x_i\}$ over Ω.

The next fact implies that the (f,g)-expansion formula of $F(x)$ is unique if it exists.

Lemma 3.2. *The coefficients $\chi(n)$ in (3.7) are uniquely determined by $\{G(i)\}$, i.e.,*

$$\chi(n) = \mathbb{D}_{(f,g)}^{(n)} \begin{bmatrix} b_0, b_1, \ldots, b_n \\ x_1, \ldots, x_{n-1} \end{bmatrix} \{F(x)\}. \tag{3.8}$$

It is worth pointing out that even if the (f,g)-series generated by $F(x)$ does not converge to itself, it always agrees with $F(x)$ at infinite points $b_i, i = 0, 1, \cdots$. Hence, in order to guarantee that they are equal over Ω, it is sufficient to require that the series and $F(x)$ be analytic around $x = b$ while b can be chosen as an accumulation point of $\{b_i\}$ in the interior of Ω. Combining this idea with the original "Ismail's argument" due to Ismail, we get all that described in the present paper.

Before proceeding to our main results, we had better display two examples which can be interpreted in this view. One is Heine's q-analogue of the Gauss summation formula (cf.[22, II.8])

Example 3.1. *For four indeterminate q, a, c, x with $|q| < 1$ and $|cx/a| < 1$,*

$${}_2\phi_1 \left[\begin{matrix} a, & 1/x \\ & c \end{matrix} ; q, \frac{cx}{a} \right] = \frac{(c/a, cx; q)_\infty}{(c, cx/a; q)_\infty}. \tag{3.9}$$

Another is the famous Rogers-Fine identity [14, p.15, Eq. (14.1)]

Example 3.2. *For three indeterminate q, x, z with $|q| < 1$ and $|z| < 1$,*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(x;q)_n} z^n = \sum_{k=0}^{\infty} (1 - azq^{2k}) q^{2\binom{k}{2}} \frac{(a;q)_k (azq/x;q)_k}{(z;q)_{k+1}(x;q)_k} (xz)^k. \tag{3.10}$$

Indeed, the former summation formula is an $(1, x-y)$-expansion formula of the function $F(x) = \frac{(c/a,cx;q)_\infty}{(c,cx/a;q)_\infty}$ on the open set $\{x : |cx/a| < 1\}$ while the latter transformation formula is an $(1 - xy, x - y)$-expansion formula of the function $F(x) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(x;q)_n} z^n$.

Whereas many nonterminating summation and transformation formulas from the theory of basic hypergeometric series, just like Examples 3.1 and 3.2, fit into such a framework, apparently there is no general theorem (aside from those formulas mentioned in Section 1) known on the existence of such phenomena. Now the definition of (f,g)-expansion formula permits us to establish the following version of the "Ismail's argument".

Theorem 3.1 (Generalized Ismail's argument)**.** *Let $F(x) \in \mathcal{H}(\Omega)$, $f(x,y) \in Ker\mathcal{L}_3^{(g)}$. Let $S_n(x)$ be the sequence of partial sums of the (f,g)-series generated by $F(x)$, say*

$$S_n(x) = \sum_{k=0}^{n} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)} \tag{3.11}$$

where the coefficients

$$G(k) = \mathbb{D}^{(k)}_{(f,g)}\begin{bmatrix} b_0, b_1, \ldots, b_k \\ x_1, \ldots, x_{k-1}\end{bmatrix}\{F(x)\}. \tag{3.12}$$

Assume further that

- **(i):** $\lim_{n\mapsto\infty} b_n = b \in \Omega$;
- **(ii):** $S_n(x)$ *converges uniformly to* $S(x)$ *in a neighborhood of* $b \in \Omega$;
- **(iii):** *for any integer* $i \geq 0, g(b_i, x)/f(x_{i+1}, x)$ *is analytic at* $x = b$.

Then there exists a subset $\Omega_1 \subseteq \Omega$ *containing* b *such that for* $x \in \Omega_1$,

$$F(x) = S(x) = \sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)}. \tag{3.13}$$

Proof. At first, we apply the (f, g)-inversion to the r.h.s. of (3.11) to arrive at

$$F(b_n) = \sum_{k=0}^{n} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, b_n)}{\prod_{i=1}^{k} f(x_i, b_n)}, \quad \text{thus,} \quad F(b_n) = S_n(b_n). \tag{3.14}$$

Taken together, the known conditions (ii)-(iii) ensures that $S(x)$ is, by Weierstrass theorem (cf.[1, p.176,Theorem 1]), analytic at $x = b$. Therefore, taking the limit $n \to \infty$ on both sides of (3.14) and using these basic relations

$$\begin{aligned}
&\lim_{n\mapsto\infty} F(b_n) = F(\lim_{n\mapsto\infty} b_n) = F(b);\\
&S_n(b_n) = \sum_{k=0}^{n} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, b_n)}{\prod_{i=1}^{k} f(x_i, b_n)} = \sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, b_n)}{\prod_{i=1}^{k} f(x_i, b_n)};\\
&\lim_{n\mapsto\infty} S_n(b_n) = \lim_{n\mapsto\infty} \sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, b_n)}{\prod_{i=1}^{k} f(x_i, b_n)}\\
&= \sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} \lim_{n\mapsto\infty} g(b_i, b_n)}{\prod_{i=1}^{k} \lim_{n\mapsto\infty} f(x_i, b_n)} = S(b),
\end{aligned}$$

we obtain $F(b) = S(b)$. Based on this, by invoking analytic continuation argument, there must exist a subset $\Omega_1 \subseteq \Omega$, such that $b \in \Omega_1$ and for $x \in \Omega_1$,

$$F(x) = \sum_{k=0}^{\infty} G(k) f(x_k, b_k) \frac{\prod_{i=0}^{k-1} g(b_i, x)}{\prod_{i=1}^{k} f(x_i, x)} = S(x).$$

So the proof is complete. ∎

Remark 3.1. *Note that the correctness of (3.13) implies that the set of functions* $\frac{\prod_{i=0}^{k-1} g(b_i,x)}{\prod_{i=1}^{k} f(x_i,x)}$ $(k = 0, 1, 2, \cdots)$ *is a basis of the linear space* $\mathcal{H}(\Omega_1)$ *of analytic functions over* Ω_1. *For a full characterization of functions* $f(x, y) \in \mathrm{Ker}\mathcal{L}_3^{(g)}$ *such that Condition (iii) holds, see* [38] *for more details.*

Remark 3.2. *Note that Conditions* (i) *and* (ii) *are also necessary. Otherwise, the conclusion may be false. For instance, let* $F(x) = \sin(\pi x)$ *and parameter sequence* $b_k = k, k = 1, 2, \cdots$. *Assume that*

$$\sin(\pi x) = \sum_{k=0}^{\infty} G(k) x(x-1)(x-2)\cdots(x-k+1).$$

By the $(1, x-y)$*-inversion, it would follow that*

$$G(n) = \mathbb{D}^{(n)}_{(x-y)}[0, 1, \ldots, n]\{\sin(\pi\, x)\} \equiv 0$$

for $n \geq 1$, *i.e.,* $\sin(\pi x) = 1$, *which is obviously contrary to the known fact.*

Remark 3.3. *As one may expect, the practical difficulty with this theorem is the verification of Condition* (ii). *In this regard, for the case of* $(1, x-y)$*-expansion formula, López-Macro -Parcet* [35, p.110, Theorem 2.6] *showed that there exists a disk* $\mathbb{O}_r = \{x : |x| < r\}$ *such that* $S_n(x)$ *converges uniformly with the requirement that* $\{b_i\}$ *be a* P*-sequence. Before this, Ismail and Stanton* (cf.[27, Theorem 3.3]) *established a* q*-Taylor theorem for any entire function being of* q*-exponential growth of order* $2c\ln^2 q$ *which is a special case of the* $(1, x-y)$*-expansion formula with respect to the sequence* $b_i = (aq^i + 1/(aq^i))/2$*. Also, the situation is greatly different when* $F(x)$ *is polynomial or a (basic) hypergeometric series. In either of these two cases, Condition* (ii) *is considerably easy to check.*

Five important cases of Theorem 3.1 are worthy of note, which are obtained by specializing $(f(x,y), g(x,y)) = (1, x-y), (x-y, x-y), (1-xy, x-y)$, and $((1-axy)(1-b\frac{x}{y}), (x-y)(1-\frac{b}{axy}))$, and $(y\theta(xy)\theta(\frac{x}{y}), y\theta(xy)\theta(\frac{x}{y}))$, respectively. We now summarize these results without proof.

Corollary 3.1. *With the same assumption as Theorem 3.1. Then the following hold*

$$F(x) = \sum_{k=0}^{\infty}\left\{\sum_{j=0}^{k}\frac{F(b_j)}{\prod_{i=0,i\neq j}^{k}(b_i-b_j)}\right\}\prod_{i=0}^{k-1}(b_i-x) \tag{3.15}$$

$$= \sum_{k=0}^{\infty}(x_k-b_k)\left\{\sum_{j=0}^{k}F(b_j)\frac{\prod_{i=1}^{k-1}(x_i-b_j)}{\prod_{i=0,i\neq j}^{k}(b_i-b_j)}\right\}\frac{\prod_{i=0}^{k-1}(b_i-x)}{\prod_{i=1}^{k}(x_i-x)} \tag{3.16}$$

$$= \sum_{k=0}^{\infty}(1-x_kb_k)\left\{\sum_{j=0}^{k}F(b_j)\frac{\prod_{i=1}^{k-1}(1-x_ib_j)}{\prod_{i=0,i\neq j}^{k}(b_i-b_j)}\right\}\frac{\prod_{i=0}^{k-1}(b_i-x)}{\prod_{i=1}^{k}(1-xx_i)} \tag{3.17}$$

$$= \sum_{k=0}^{\infty}G_1(k)(1-ax_kb_k)(1-bx_k/b_k)\frac{\prod_{i=0}^{k-1}(b_i-x)(1-\frac{b}{ab_ix})}{\prod_{i=1}^{k}(1-ax_ix)(1-b\frac{x_i}{x})}; \tag{3.18}$$

$$= \sum_{k=0}^{\infty}G_2(k)b_k\theta(x_kb_k)\theta\left(\frac{x_k}{b_k}\right)\frac{\prod_{i=0}^{k-1}\theta(b_ix)\theta(\frac{b_i}{x})}{\prod_{i=1}^{k}\theta(x_ix)\theta(\frac{x_i}{x})}, \tag{3.19}$$

where the coefficients

$$G_1(k) = \mathbb{D}^{(k)}_{\left((1-axy)(1-b\frac{x}{y}),(x-y)(1-\frac{b}{axy})\right)}\begin{bmatrix}b_0, b_1, \ldots, b_k\\ x_1, \ldots, x_{k-1}\end{bmatrix}\{F(x)\},$$

$$G_2(k) = \mathbb{D}^{(k)}_{\left(y\theta(xy)\theta(\frac{x}{y}),y\theta(xy)\theta(\frac{x}{y})\right)}\begin{bmatrix}b_0, b_1, \ldots, b_k\\ x_1, \ldots, x_{k-1}\end{bmatrix}\{F(x)\}, \qquad \textit{and}$$

$\theta(x) = (x;q)_\infty(\frac{q}{x};q)_\infty$, $|q| < 1$.

We remark that the theta function $\theta(x)$ satisfying Jacobi's *triple product identity*

$$\theta(x)(q;q)_\infty = \sum_{k=-\infty}^{\infty}(-1)^k q^{\binom{k}{2}}x^k, x \neq 0$$

has been frequently used in the study of the elliptic hypergeometric series. See [47, 50] for more details.

In spite of lacking a general way to verify, as pointed out in Remark 3.3, the uniform convergency of $S_n(x)$, we have yet the following positive result for the expansion formula (3.17).

Theorem 3.2 (The $(1-xy, x-y)$-expansion theorem)**.** *Let* $F(x) \in \mathcal{H}(\Omega)$*,* $\{b_i\}, \{x_i\} \subseteq \Omega$ *such that* $\{b_i\}$ *are pairwise distinct and* $\{x_i\}$ *is bounded,* $\lim_{i\mapsto\infty} b_i = b \neq b_0$ *and* $\inf\{|1/x_i - b| : i \geq 0\} > 0$*. Suppose further that* $\limsup_{k\mapsto\infty}|\lambda_k| < \infty$ *where*

$$\lambda_k = \mathbb{D}^{(k)}_{(1-xy,x-y)}\begin{bmatrix}b_1, b_2, \ldots, b_{k+1}\\ x_1, \ldots, x_{k-1}\end{bmatrix}\{F(x)\}/\,\mathbb{D}^{(k)}_{(1-xy,x-y)}\begin{bmatrix}b_0, b_1, \ldots, b_k\\ x_1, \ldots, x_{k-1}\end{bmatrix}\{F(x)\}. \tag{3.20}$$

Then there exists an open set Ω_1 *containing* b *such that for* $x \in \Omega_1$,

$$F(x)=\sum_{k=0}^{\infty}(1-x_kb_k)\left\{\mathbb{D}^{(k)}_{(1-xy,x-y)}\begin{bmatrix}b_0,b_1,\ldots,b_k\\ x_1,\ldots,x_{k-1}\end{bmatrix}\{F(x)\}\right\}\frac{\prod_{i=0}^{k-1}(b_i-x)}{\prod_{i=1}^{k}(1-xx_i)}. \tag{3.21}$$

Proof. Write r_0 for $\inf\{|1/x_i-b|: i\geq 0\}$. The assumption that $r_0>0$ means that each function $1/(1-xx_i)$ has no pole in the disk $\mathbb{O}_{r_0}=\{x:|x-b|<r_0\}$. So according to Theorem 3.1, it needs only to check that there exists an open set Ω_1 containing b , such that $S_n(x)\mapsto S(x)$ uniformly on $\Omega_1\subseteq\mathbb{O}_{r_0}$. For notational simplicity, we write

$$G(k)=\mathbb{D}^{(k)}_{(1-xy,x-y)}\begin{bmatrix}b_0,b_1,\ldots,b_k\\ x_1,\ldots,x_{k-1}\end{bmatrix}\{F(x)\}.$$

It is clear that by the recursive formula (2.5) for the (f,g)-difference operator, the ratio

$$\left|\frac{G(k+1)}{G(k)}\right| = \left|\frac{1-x_kb_0}{b_0-b_{k+1}}+\lambda_k\frac{1-x_kb_{k+1}}{b_{k+1}-b_0}\right|.$$

Consider that

$$\frac{1-x_kb_0}{b_0-b_{k+1}}+\lambda_k\frac{1-x_kb_{k+1}}{b_{k+1}-b_0}=x_k+(\lambda_k-1)\frac{1-x_kb_{k+1}}{b_{k+1}-b_0}.$$

Thus, by the triangular inequality, we have

$$\left|\frac{G(k+1)}{G(k)}\right| \leq |x_k|+|\lambda_k-1|\left|\frac{1-x_kb_{k+1}}{b_{k+1}-b_0}\right|. \tag{3.22}$$

So, given a constant $M_1=1+\max\{m,(1+|b|m)/|b-b_0|\}$, where $m=\sup\{|x_k|:k\geq 0\}$, by solving the inequality under the known condition that $\lim_{i\mapsto\infty}b_i=b$, we can prove there exists such an integer K_1 that for $k>K$,

$$\left|\frac{1-x_kb_{k+1}}{b_{k+1}-b_0}\right|<M_1$$

provided that $M_1\geq\max\{m,(1+m|b|)/|b-b_0|\}$. Next, let $M=m+m_0M_1$, $m_0=1+\limsup_{k\mapsto\infty}|\lambda_k|$. So for any $k>K$,

$$\left|\frac{G(k+1)}{G(k)}\right|\leq|x_k|+|\lambda_k-1|\left|\frac{1-x_kb_{k+1}}{b_{k+1}-b_0}\right|<m+m_0M_1=M. \tag{3.23}$$

Finally, decide a constant $s<r_0$ by solving the following inequality for the fixed M and a chosen $a:0<a<1$, s is independent of k and K, such that $|x-b|<s$ small enough, in order to get

$$\left|\frac{1-x_{k+1}b_{k+1}}{1-x_kb_k}\frac{b_k-x}{1-x_kx}\right|<\frac{a}{M}\left(<\frac{1}{m}\right). \tag{3.24}$$

Finally, Inequalities (3.23) and (3.23) together states that for $k>K$,

$$\left|\frac{(k+1)\text{-th term of (3.21)}}{k\text{-th term of (3.21)}}\right|=\left|\frac{G(k+1)}{G(k)}\frac{1-x_{k+1}b_{k+1}}{1-x_kb_k}\frac{b_k-x}{1-x_kx}\right|<M\times\frac{a}{M}=a<1.$$

By the ratio test, we see that $S_n(x)$ indeed converges uniformly on the disk $\mathbb{O}_s=\{x:|x-b|<s\}$, which is such a required open set Ω_1. Finally, the desired follows from Theorem 3.1. ∎

Notice that when $F(x)$ is a basic hypergeometric series, it holds that $\lim_{k\mapsto\infty}|\lambda_k|=1$. It is also worthy of note that Fu and Lascoux, in their paper [16], established a similar expansion formula in the setting of formal power series by virtue of a divided difference operator acting on multivariate function. For a discussion of this divided difference operator and related matters, the reader may consult [33].

4. An analytic proof of Gessel and Stanton's q-analogue of the Lagrange inversion formula

Actually, Theorem 3.2 contains as specifical cases the q-analogue of the Lagrange inversion formula due to Gessel and Stanton (see Lemma 7.2 in Appendix for a proof of this previously unknown fact), Liu's expansion formula, and Carlitz's q-analogue. We obtain them only by specializing $b_i = q^i, x_i = Ap^i$ in (3.21). Certainly, as initial study on the representation problem of $F(x)$ in terms of (f,g)-series, any possibly rigorous analytic proof of these formulas is worth investigating. As a result, such a proof is obtained as follows.

Theorem 4.1. *Let $|p|,|q| < 1$ and $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be arbitrary power series with the nonzero radius R of convergence and be also convergent at $x = R$. Assume further R is not of the form $q^n (n \geq 0)$, $\lim_{n\mapsto\infty} a_{n+1}/a_n = c_0$, and $0 < m|c_0|^n \leq |a_n| \leq M|c_0|^n$. Then*

$$\begin{aligned} F(x) &= \sum_{k=0}^{\infty} G_3(k)(1-Ap^kq^k)\frac{\prod_{i=0}^{k-1}(q^i-x)}{(q;q)_k(Apx;p)_k} \\ &\qquad\qquad\qquad\qquad (\Leftrightarrow \text{Gessel and Stanton's } q\text{-analogue } (1.3)/(1.4)) \\ &= \sum_{k=0}^{\infty} G_4(k)\frac{(1-aq^{2k})(aq/x;q)_k x^k}{(q;q)_k(x;q)_k} \qquad (\text{Liu's } q\text{-expansion formula } (1.5)) \\ &= \sum_{k=0}^{\infty} \frac{x^k}{(q,x;q)_k}\left[\mathcal{D}_{q,x}^k\{F(x)(x;q)_{k-1}\}\right]_{x=0} \qquad (\text{Carlitz's } q\text{-analogue } (1.2)) \end{aligned}$$

where the coefficients

$$G_3(n) = \sum_{k=0}^{n}(-1)^{n-k}q^{\binom{k+1}{2}-nk}\begin{bmatrix} n \\ k \end{bmatrix}_q (Apq^k;p)_{n-1}F(q^k); \tag{4.1}$$

$$G_4(n) = \frac{1}{a^nq^n}\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-nk}\begin{bmatrix} n \\ k \end{bmatrix}_q (aq^{k+1};q)_{n-1}F(aq^{k+1}). \tag{4.2}$$

Proof. Since Liu's q-expansion formula (1.5) follows from the q-analogue of Gessel and Stanton by substituting $p \mapsto q, A \mapsto a, x \mapsto x/aq$, and Identity (1.2) is the special case $a \mapsto 0$ of (1.5) together with Proposition 2.2, it needs only to show the q-analogue of Gessel and Stanton. To do this, suggested by Theorem 3.2, it suffices to check that $\lim_{n\mapsto\infty}|\lambda_n| < \infty$, because that $b = 0, r_0 = |1/A|$. Here, it is convenient to rewrite λ_n in a compact form

$$\lambda_n = \mathbb{L}_{1:n+1}\{F(x)\}/\mathbb{L}_{0:n}\{F(x)\} \tag{4.3}$$

with two difference operators employed

$$\begin{aligned} \mathbb{L}_{0:n}\{\bullet\} &= \mathbb{D}^{(n)}_{(1-xy,x-y)}\begin{bmatrix} 1,q,q^2,\ldots,q^n \\ Ap,Ap^2,\ldots,Ap^{n-1} \end{bmatrix}\{\bullet\}; \\ \mathbb{L}_{1:n+1}\{\bullet\} &= \mathbb{D}^{(n)}_{(1-xy,x-y)}\begin{bmatrix} q,q^2,q^3,\ldots,q^{n+1} \\ Ap,Ap^2,\ldots,Ap^{n-1} \end{bmatrix}\{\bullet\}. \end{aligned}$$

Our consideration breaks up into the following two cases.

Case I. At first, let $F(x) = x^r$. By using of the fact

$$\sum_{k=0}^{n}\frac{b_k^r}{\prod_{i=0,i\neq k}^{n}(b_i-b_k)} = (-1)^n h_{r-n}(b_0,b_1,b_2,\cdots,b_n), \tag{4.4}$$

it is easily found that

$$\mathbb{D}^{(n)}_{(1-xy,x-y)}\begin{bmatrix}b_0,b_1,b_2,\cdots,b_n\\ x_1,x_2,\cdots,x_{n-1}\end{bmatrix}\{x^r\}=\sum_{k=0}^{n-1}(-1)^{n-k}e_k(x_1,x_2,\cdots,x_{n-1})h_{r+k-n}(b_0,b_1,b_2,\cdots,b_n),$$

where h_i (resp. e_i), as the ith complete symmetric function of $b_0,b_1,\cdots,b_n$ (resp. $x_1,x_2,\cdots,x_{n-1}$), is given by

$$h_r(b_0,b_1,b_2,\cdots,b_n)=\sum_{0\le i_1\le i_2\le\cdots\le i_r\le n}b_{i_1}b_{i_2}\cdots b_{i_r};$$
$$e_k(x_1,x_2,\cdots,x_{n-1})=\sum_{1\le i_1<i_2<\cdots<i_k\le n-1}x_{i_1}x_{i_2}\cdots x_{i_k}.$$

With the choices that $b_i=q^i, x_i=Ap^i$, we establish by induction on r (resp. k) that

$$h_r(1,q,q^2,\cdots,q^n)=\sum_{0\le i_1\le i_2\le\cdots\le i_r\le n}q^{i_1+i_2+\cdots+i_r}=\begin{bmatrix}n+r\\ n\end{bmatrix}_q;$$
$$e_k(Ap,Ap^2,\cdots,Ap^{n-1})=\sum_{1\le i_1<i_2<\cdots<i_k\le n-1}A^kp^{i_1+i_2+\cdots+i_k}=A^kp^{\binom{k+1}{2}}\begin{bmatrix}n-1\\ k\end{bmatrix}_p.$$

These results combined together leads us to

$$\begin{aligned}\mathbb{L}_{0:n}\{x^r\}&=\sum_{k=0}^{n-1}(-1)^{n-k}A^kp^{\binom{k+1}{2}}\begin{bmatrix}n-1\\ k\end{bmatrix}_p\begin{bmatrix}r+k\\ n\end{bmatrix}_q\\&=\sum_{k=1}^{r}(-1)^kA^{n-k}p^{\binom{n-k+1}{2}}\begin{bmatrix}n-1\\ n-k\end{bmatrix}_p\begin{bmatrix}r+n-k\\ r-k\end{bmatrix}_q;\end{aligned}$$

$$\begin{aligned}\mathbb{L}_{1:n+1}\{x^r\}&=\sum_{k=0}^{n-1}(-1)^{n-k}A^kp^{\binom{k+1}{2}}\begin{bmatrix}n-1\\ k\end{bmatrix}_pq^{r+k-n}\begin{bmatrix}r+k\\ n\end{bmatrix}_q\\&=\sum_{k=1}^{r}(-1)^kA^{n-k}p^{\binom{n-k+1}{2}}q^{r-k}\begin{bmatrix}n-1\\ n-k\end{bmatrix}_p\begin{bmatrix}r+n-k\\ r-k\end{bmatrix}_q.\end{aligned}$$

Finally, we calculate directly

$$\begin{aligned}\lim_{n\mapsto\infty}\lambda_n=\lim_{n\mapsto\infty}\frac{\mathbb{L}_{1:n+1}\{x^r\}}{\mathbb{L}_{0:n}\{x^r\}}&=\lim_{n\mapsto\infty}\frac{\sum_{k=1}^r(-1)^{n-k}A^{n-k}p^{\binom{n-k+1}{2}}q^{r-k}\left[\begin{smallmatrix}n-1\\ k-1\end{smallmatrix}\right]_p\left[\begin{smallmatrix}r+n-k\\ r-k\end{smallmatrix}\right]_q}{\sum_{k=1}^r(-1)^{n-k}A^{n-k}p^{\binom{n-k+1}{2}}\left[\begin{smallmatrix}n-1\\ n-k\end{smallmatrix}\right]_p\left[\begin{smallmatrix}r+n-k\\ r-k\end{smallmatrix}\right]_q}\\&=\frac{\sum_{k=1}^r\lim_{n\mapsto\infty}(-1)^kA^{-k}p^{n(r-k)+\binom{k}{2}}q^{r-k}\left[\begin{smallmatrix}n-1\\ k-1\end{smallmatrix}\right]_p\left[\begin{smallmatrix}r+n-k\\ r-k\end{smallmatrix}\right]_q}{\sum_{k=1}^r\lim_{n\mapsto\infty}(-1)^kA^{-k}p^{n(r-k)+\binom{k}{2}}\left[\begin{smallmatrix}n-1\\ k-1\end{smallmatrix}\right]_p\left[\begin{smallmatrix}r+n-k\\ r-k\end{smallmatrix}\right]_q}=1\end{aligned}\tag{4.5}$$

by noting that

$$\lim_{n\mapsto\infty}(-1)^kA^{-k}p^{n(r-k)+\binom{k}{2}}\begin{bmatrix}n-1\\ k-1\end{bmatrix}_p\begin{bmatrix}r+n-k\\ r-k\end{bmatrix}_q=\begin{cases}0, & k\neq r;\\ (-1)^rA^{-r}p^{\binom{r}{2}}\frac{1}{(p;p)_{r-1}}, & k=r.\end{cases}$$

Case II. Next, suppose that $F(x)=\sum_{r=0}^{\infty}a_rx^r$ is a power series with the nonzero radius R of convergency. By the definition, we have

$$\begin{aligned}\mathbb{L}_{1:n+1}\{F(x)\}&=\sum_{k=1}^{n}(-1)^kA^{n-k}p^{\binom{n-k+1}{2}}\begin{bmatrix}n-1\\ n-k\end{bmatrix}_p\sum_{r=k}^{\infty}a_rq^{r-k}\begin{bmatrix}r+n-k\\ r-k\end{bmatrix}_q\\&=\sum_{k=1}^{n}(-1)^kA^{n-k}p^{\binom{n-k+1}{2}}\begin{bmatrix}n-1\\ n-k\end{bmatrix}_p\sum_{r=0}^{\infty}a_{r+k}q^{r}\begin{bmatrix}r+n\\ r\end{bmatrix}_q.\end{aligned}\tag{4.6}$$

For the sake of simplicity, define that

$$K_{n,k}(x) \quad = \quad \sum_{r=0}^{\infty} a_{r+k} \begin{bmatrix} r+n \\ r \end{bmatrix}_q x^r. \tag{4.7}$$

By this definition, it is easily found that the generating function of $K_{n,k}(x)$ in $t : |t| < 1$

$$\sum_{k=0}^{\infty} K_{n,k}(x) t^k = \frac{F(t)}{(x/t;q)_{n+1}} \qquad \text{and} \tag{4.8}$$

$$\mathcal{D}_{q,x}^n\{F(x)\} = (q;q)_n K_{n,n}(x). \tag{4.9}$$

Further, from the known condition $\lim_{n\mapsto\infty} a_{n+1}/a_n = c_0$, it follows

$$\lim_{n\mapsto\infty} \frac{K_{n,n}(x)}{a_n} = \sum_{r=0}^{\infty} \lim_{n\mapsto\infty} \frac{a_{r+n}}{a_n} \begin{bmatrix} r+n \\ r \end{bmatrix}_q x^r = \sum_{r=0}^{\infty} \frac{(xc_0)^r}{(q;q)_r} = \frac{1}{(xc_0;q)_\infty}. \tag{4.10}$$

Note that, by Hadamard's formula, $|c_0| = 1/R$. Two recursive relations of interest implied by (4.8) are

$$\begin{aligned} K_{n,k}(x) - K_{n,k}(qx) &= xK_{n,k+1}(x) - xq^{n+1}K_{n,k+1}(qx); \\ K_{n,k}(x) &= \frac{1}{1-q^n} K_{n-1,k}(x) - \frac{q^n}{1-q^n} K_{n-1,k}(qx). \end{aligned}$$

By iterating the last relation m times, we obtain at once

$$K_{n,k}(x) \quad = \quad \frac{(q;q)_{n-m}}{(q;q)_n} \sum_{i=0}^{m} (-1)^i q^{(n-m+1)i+\binom{i}{2}} \begin{bmatrix} m \\ i \end{bmatrix}_q K_{n-m,k}(q^i x). \tag{4.11}$$

Next, replacing k by $n-k$ and letting $m = k$ simultaneously in (4.11) gives

$$K_{n,n-k}(x) \quad = \quad \frac{(q;q)_{n-k}}{(q;q)_n} \sum_{i=0}^{k} (-1)^i q^{(n-k+1)i+\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q K_{n-k,n-k}(q^i x).$$

Reformulate by this fact the sum

$$\begin{aligned} &\sum_{k=1}^{n} (-1)^k A^{n-k} p^{\binom{n-k+1}{2}} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_p K_{n,k}(x) = \sum_{k=0}^{n-1} (-1)^{n-k} A^k p^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_p K_{n,n-k}(x) \\ = \quad &\sum_{k=0}^{n-1} (-1)^{n-k} A^k p^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_p \frac{(q;q)_{n-k}}{(q;q)_n} \sum_{i=0}^{k} (-1)^i q^{(n-k+1)i+\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q K_{n-k,n-k}(q^i x). \end{aligned}$$

Now, we turn to the limitation of $\mathbb{L}_{1:n+1}\{F(x)\}/\mathbb{L}_{0:n}\{F(x)\}$ as $n \mapsto \infty$. For this we may split $\mathbb{L}_{1:n+1}\{F(x)\}$ and $\mathbb{L}_{0:n}\{F(x)\}$ into

$$\begin{aligned} \mathbb{L}_{1:n+1}\{F(x)\} &= (-1)^n K_{n,n}(q) + S(n,q) \qquad \text{and} \\ \mathbb{L}_{0:n}\{F(x)\} &= (-1)^n K_{n,n}(1) + S(n,1) \end{aligned}$$

by defining

$$S(n,x) \quad = \quad \sum_{k=1}^{n-1} (-1)^{n-k} A^k p^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_p \frac{(q;q)_{n-k}}{(q;q)_n} \sum_{i=0}^{k} (-1)^i q^{(n-k+1)i+\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q K_{n-k,n-k}(q^i x).$$

By using of Tannery's theorem (cf.[6] or [13, Appendix: Tannery's limiting theorem]), we obtain that $\lim_{n\mapsto\infty} S(n,x)/a_n = 0$ [1] while $x = 1, q$. This leads us to

$$\begin{aligned} \lim_{n\mapsto\infty} |\lambda_n| \quad &= \quad \lim_{n\mapsto\infty} \left| \frac{\mathbb{L}_{1:n+1}\{F(x)\}}{\mathbb{L}_{0:n}\{F(x)\}} \right| = \lim_{n\mapsto\infty} \left| \frac{(-1)^n K_{n,n}(q)/a_n + S(n,q)/a_n}{(-1)^n K_{n,n}(1)/a_n + S(n,1)/a_n} \right| \\ &= \quad \lim_{n\mapsto\infty} \left| \frac{(-1)^n K_{n,n}(q)/a_n}{(-1)^n K_{n,n}(1)/a_n} \right| = \left| \frac{(c_0;q)_\infty}{(qc_0;q)_\infty} \right| = |1-c_0| \le 1 + 1/R. \end{aligned}$$

[1]For its proof, see Lemma 7.1 in Appendix.

The latter arises from (4.10). This gives the complete proof of theorem. ∎

This corollary states clearly that, just as mentioned earlier, the $(1-xy, x-y)$-expansion formula (3.21) indeed unifies Gessel and Stanton's q-analogue, Liu's expansion formula, and Carlitz's q-analogue of the Lagrange inversion formula. The approach we have presented as above is different from that of Liu [34], who established (1.5) by means of (1.2) and expressed the coefficients $G_2(k)$ in terms of $\mathcal{D}^i_{q,x}, i=0,1,2,...,$ which is also different from (4.2).

As an illustrative example, the calculation of $\mathbb{L}_{0:n}\{\frac{1}{1-cx}\}$ ($c\neq 0$) leads us to an interesting expansion formula.

Example 4.3. *Let $F(x)=\frac{1}{1-cx}$, $|cx|<1, c\neq 0$. In this case, we calculate by the definition*

$$\begin{aligned}
\mathbb{L}_{0:n}\{\frac{1}{1-cx}\} &= \sum_{r=0}^{\infty} c^r \mathbb{L}_{0:n}\{x^r\} = \sum_{r=0}^{\infty} c^r \sum_{k=1}^{r} (-1)^k A^{n-k} p^{\binom{n-k+1}{2}} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_p \begin{bmatrix} r+n-k \\ r-k \end{bmatrix}_q \\
&= \sum_{k=1}^{n} (-1)^k A^{n-k} p^{\binom{n-k+1}{2}} \begin{bmatrix} n-1 \\ n-k \end{bmatrix}_p \sum_{r=k}^{\infty} c^r \begin{bmatrix} r+n-k \\ r-k \end{bmatrix}_q \\
&= \frac{(-1)^n c^n}{(c;q)_{n+1}} \sum_{k=0}^{n-1} (-1)^k (Ap/c)^k p^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_p = (-1)^n \frac{c^n (Ap/c;p)_{n-1}}{(c;q)_{n+1}}.
\end{aligned}$$

Similarly, we have

$$\mathbb{L}_{1:n+1}\{\frac{1}{1-cx}\} = \frac{(-1)^n c^n}{(cq;q)_{n+1}} \sum_{k=0}^{n-1} (-1)^k (Ap/c)^k p^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_p = (-1)^n \frac{c^n (Ap/c;p)_{n-1}}{(cq;q)_{n+1}}.$$

This not only gives that $\lim_{n\mapsto\infty}\lambda_n = 1-c$ but also leads to the expansion formula

$$\frac{1}{1-cx} = \frac{1}{(1-c)(1-A/c)} \sum_{k=0}^{\infty} \frac{1-Ap^k q^k}{(q;q)_k} \frac{(A/c;p)_k}{(cq;q)_k} \frac{(1/x;q)_k}{(Apx;p)_k} (cx)^k. \tag{4.12}$$

The special case $A=0$ of this result gives a refined formula

$$\frac{1-c}{1-cx} = \sum_{k=0}^{\infty} \frac{(1/x;q)_k}{(q,cq;q)_k} (cx)^k. \tag{4.13}$$

Its equivalent form given by (4.1) is

$$\frac{c^n}{(c;q)_{n+1}} = \frac{1}{(q;q)_n} \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{1-cq^k}. \tag{4.14}$$

Restate it in terms of $\mathcal{D}_{q,x}$. The result is

$$\mathcal{D}^n_{q,x}\left\{\frac{1}{1-cx}\right\}|_{x=1} = \frac{c^n (q;q)_n}{(c;q)_{n+1}}. \tag{4.15}$$

5. Applications to basic hypergeometric series

Our applications of all results in the forgoing sections to basic hypergeometric series is realized two-fold way. First, start with a given terminating summation and a properly chosen parameter sequence $b_n \mapsto b$, we want to extend it to a nonterminating summation as an (f,g)-expansion formula of certain analytic function, proceeding as the "Ismail's argument", by analytic continuity on the given region. Second, just the reverse of the above procedure, for any known (f,g)-expansion of $F(x)$, there always exists a terminating summation formula, i.e., the expression of the n-th order (f,g)-difference $G(n)$ of $F(x)$ in terms of $F(x)|_{x=b_i}$. For the limitation of space, we only consider a few remarkable summation and transformation formulas.

5.1. **From terminating summations to nonterminating summations.** Since it is the origin of the "Ismail's argument", we would like to adopt Ismail's proof of Ramanujan's ${}_1\varphi_1$ summation formula as our first example. See [25] or [3, pp.504-505] for more details.

Theorem 5.1. *For $|q|<1$ and $|ba^{-1}|<|x|<1$,*

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(b;q)_n}x^n=\frac{(q,b/a,ax,q/(ax);q)_\infty}{(b,q/a,x,b/(ax);q)_\infty}. \tag{5.1}$$

Proof. At first, define two functions of a variable y

$$S(y) = \sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(y;q)_n}x^n \qquad\text{and}\quad F(y)=\frac{(q,y/a,ax,q/(ax);q)_\infty}{(y,q/a,x,y/(ax);q)_\infty}.$$

Evidently, $F(y)$ and $S(y)$ are analytic around $y=0=\lim_{N\mapsto\infty}b_N,\ b_N=q^{N+1}, N\geq 0$. And then to show $F(b_N)=S(b_N)$ for each N, i.e.,

$$\sum_{n=-\infty}^{\infty}\frac{(a;q)_n}{(q^{N+1};q)_n}x^n=\frac{(q,q^{N+1}/a,ax,q/(ax);q)_\infty}{(q^{N+1},q/a,x,q^{N+1}/(ax);q)_\infty}, \tag{5.2}$$

which follows from rewriting the q-binomial theorem as follows

$$\begin{aligned}\frac{(ax;q)_\infty}{(x;q)_\infty} &= \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n=\sum_{n=-N}^{\infty}\frac{(a;q)_{n+N}}{(q;q)_{n+N}}x^{n+N}\\ &= \sum_{n=-N}^{\infty}\frac{(a;q)_{n+N}}{(q;q)_{n+N}}x^{n+N}=\frac{(a;q)_N}{(q;q)_N}x^N\sum_{n=-N}^{\infty}\frac{(aq^N;q)_n}{(q^{N+1};q)_n}x^n\end{aligned} \tag{5.3}$$

and then replacing a with aq^{-N} in the above identity. Note that by the convention or the inverse identity (cf.[22, I.2])

$$\frac{1}{(q^{N+1};q)_n}=0 \qquad\text{for}\quad n<-N-1.$$

By analytic continuation, we see that the result is true. ∎

Another example of the $(1-xy,x-y)$-expansion formula, also a good case to illustrate the "Ismail's argument", is the following generalized Lebesgue identity due to Carlitz [10].

Theorem 5.2. *For three indeterminate q,x,a with $|q|<1$ and $|x|<1$,*

$$\sum_{k=0}^{\infty}(-1)^kq^{\binom{k}{2}}\frac{(x;q)_ka^k}{(q,bx;q)_k}=\frac{(a,x;q)_\infty}{(bx;q)_\infty}\sum_{k=0}^{\infty}\frac{(b;q)_k}{(q,a;q)_k}x^k. \tag{5.4}$$

Proof. At first, apply the substitution $b\mapsto b/y, x\mapsto xy, a\mapsto y$ to reformulate equivalently the identity in question as

$$F(y)=\sum_{k=0}^{\infty}\frac{(-1)^kx^k}{(q;q)_k}\frac{\prod_{i=0}^{k-1}g(b_i,y)}{\prod_{i=1}^{k}f(x_i,y)}, \tag{5.5}$$

where $f(x,y)=1-xy, g(x,y)=x-y, b_i=bq^i, x_i=q^{i-1}$, and $F(y)$ is defined by

$$F(y)=\sum_{i=0}^{\infty}\frac{(-1)^iq^{\binom{i}{2}}(bxq^i;q)_\infty}{(q;q)_i}\frac{y^i}{(y,xyq^i;q)_\infty}. \tag{5.6}$$

By the $(1-xy,x-y)$-expansion formula, it suffices to show in a direct way that

$$\frac{(bx)^n}{1-bq^{2n-1}}=\sum_{k=0}^{n}(-1)^kq^{\binom{k+1}{2}-nk}\begin{bmatrix}n\\k\end{bmatrix}_q(bq^k;q)_{n-1}F(bq^k). \tag{5.7}$$

For this, we calculate by using Heine's transformation formula to obtain

$$
\begin{aligned}
F(bq^k) &= \frac{(bx;q)_k}{(bq^k;q)_\infty}\sum_{i=0}^{\infty}\frac{(-1)^i q^{\binom{i}{2}}(bq^k)^i}{(q;q)_i}\frac{(bxq^k;q)_i}{(bx;q)_i}\\
&= \sum_{i=0}^{\infty}\frac{(q^{-k};q)_i(bxq^k)^i}{(q,bq^k;q)_i}=\sum_{i=0}^{k}(-1)^i q^{\binom{i}{2}}\begin{bmatrix}k\\ i\end{bmatrix}_q\frac{(bx)^i}{(bq^k;q)_i}.
\end{aligned}
$$

Inserting this into the r.h.s. of (5.7) to simplify the resulting identity, we have

$$
\begin{aligned}
\text{RHS of } (5.7) &= \sum_{n\geq k\geq i\geq 0}(-1)^{k+i}q^{\binom{k+1}{2}-nk+\binom{i}{2}}\begin{bmatrix}n\\ k\end{bmatrix}_q\begin{bmatrix}k\\ i\end{bmatrix}_q\frac{(bq^k;q)_{n-1}(bx)^i}{(bq^k;q)_i}\\
&= \sum_{i=0}^{n}(bx)^i q^{-Ni}\begin{bmatrix}n\\ i\end{bmatrix}_q\sum_{K=0}^{N}(-1)^K q^{\binom{K+1}{2}-NK}\begin{bmatrix}N\\ K\end{bmatrix}_q\frac{(b;q)_{N+K+2i-1}}{(b;q)_{K+2i}},
\end{aligned}
$$

where $N=n-i, K=k-i$. Evidently, if $N\geq 1$, then

$$
\frac{(b;q)_{N+K+2i-1}}{(b;q)_{K+2i}}=(bq^{K+2i};q)_{N-1}.
$$

Since for fixed i, $(bq^{2i}x;q)_{N-1}$ is a polynomial of degree no more than $N-1$ in x, thus by Proposition 2.3, we find that the inner sum turns out to be

$$
\sum_{K=0}^{N}(-1)^K q^{\binom{K+1}{2}-NK}\begin{bmatrix}N\\ K\end{bmatrix}_q\frac{(b;q)_{N+K+2i-1}}{(b;q)_{K+2i}}=0.
$$

Thus, the only nonzero term corresponding to $N=0$ of the r.h.s. of (5.7) is $(bx)^n/(1-bq^{2n-1})$. It gives the complete proof of (5.7). ∎

Next, by the (f,g)-inversion and the q-Pfaff-Saalschütz formula (which makes our proofs given here different from all known ones), together with the "Ismail's argument", we obtain new proofs of Rogers' nonterminating very-well-poised (in short, VWP-balanced) ${}_6\phi_5$ summation formula (cf.[22, Eq.(II.20)]) and Watson's q-analogue of Whipple's transformation formula from ${}_8\phi_7$ to ${}_4\phi_3$ (cf.[22, III.17]).

Theorem 5.3 (Rogers' nonterminating VWP-balanced ${}_6\phi_5$ summation formula)**.** *For all parameters a,b,c,d where the series converges,*

(5.8)

$$
{}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d\\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d\end{matrix};q,\frac{aq}{bcd}\right]=\frac{(aq,aq/(cd),aq/(bd),aq/(bc);q)_\infty}{(aq/b,aq/c,aq/d,aq/(bcd);q)_\infty}.
$$

Proof. Note that the q-Pfaff-Saalschütz formula

$$
{}_3\phi_2\left[\begin{matrix} q^{-n}, & aq^n, & aq/bc\\ & aq/b, & aq/c\end{matrix};q,q\right]=\left(\frac{aq}{bc}\right)^n\frac{(b,c;q)_n}{(aq/b,aq/c;q)_n} \tag{5.9}
$$

can be reformulated as

$$
\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}\frac{(aq^n;q)_k}{(aq;q)_k}q^k\frac{(aq,aq/(bc);q)_k}{(aq/b,aq/c;q)_k}=\frac{(b,c;q)_n}{(aq/b,aq/c;q)_n}\left(\frac{aq}{bc}\right)^n. \tag{5.10}
$$

Here, we will invoke the following $(x-y,x-y)$-inversion originally due to Carlitz (cf.[11] or [37, Corollary 1])

$$
\left(\frac{(q^{-n};q)_k}{(q;q)_k}\frac{(aq^n;q)_k}{(aq;q)_k}q^k\right)^{-1}=\left(\frac{(q^{-n};q)_k}{(aq^{1+n};q)_k}\frac{(a;q)_k}{(q;q)_k}\frac{1-aq^{2k}}{1-a}q^{kn}\right).
$$

Apply this inversion to (5.10) to get

$$ {}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq^{1+n} \end{matrix}; q, \frac{aq^{1+n}}{bc}\right] = \frac{(aq, aq/(bc); q)_n}{(aq/b, aq/c; q)_n}. \tag{5.11} $$

Define further

$$ \begin{aligned} S(x) &= {}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & 1/x \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aqx \end{matrix}; q, \frac{aqx}{bc}\right]; \\ F(x) &= \frac{(aq, aq/(bc); q)_\infty}{(aq/b, aq/c; q)_\infty}\frac{(aqx/c, aqx/b; q)_\infty}{(aqx, aqx/(bc); q)_\infty}. \end{aligned} $$

Evidently, $F(x)$ and $S(x)$ are analytic around $x = 0$. Choose that $b_n = q^n, x_n = aq^n$. Note that $f(x_n, b_n) = 1 - aq^{2n}, \lim_{n\mapsto\infty} b_n = 0$, and $F(b_n) = S(b_n)$. Finally, by Theorem 4.1, the claimed follows. ∎

Theorem 5.4 (Watson's q-analogue of Whipple's transformation formula from ${}_8\phi_7$ to ${}_4\phi_3$)**.**

$$ \begin{aligned} &{}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & f \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f \end{matrix}; q, \frac{a^2q^2}{bcdef}\right] \\ &= \frac{(aq, aq/(ef), aq/(df), aq/(de); q)_\infty}{(aq/d, aq/e, aq/f, aq/(def); q)_\infty}\, {}_4\phi_3\left[\begin{matrix} aq/(bc), & d, & e, & f \\ & def/a, & aq/b, & aq/c \end{matrix}; q, q\right] \end{aligned} \tag{5.12} $$

provided the ${}_4\phi_3$ *series terminates.*

Proof. By the q-Pfaff-Saalschütz formula, we obtain

$$ {}_3\phi_2\left[\begin{matrix} q^{-n}, & aq^n, & aq/bc \\ & aq/b, & aq/c \end{matrix}; q, q\right] = \left(\frac{aq}{bc}\right)^n \frac{(b, c; q)_n}{(aq/b, aq/c; q)_n}. $$

Multiply both sides of this identity by $\left(\frac{aq}{de}\right)^n \frac{(d,e;q)_n}{(aq/d,aq/e;q)_n}$ and reformulate the resulting identity to arrive at

$$ \begin{aligned} &\sum_{k=0}^{n} \frac{(q^{-n}, aq^n, aq/bc, d, e; q)_k}{(q, aq/b, aq/c, aq/d, aq/e; q)_k} q^k \left(\frac{aq}{de}\right)^n \frac{(dq^k, eq^k; q)_{n-k}}{(aq^{1+k}/d, aq^{1+k}/e; q)_{n-k}} \\ &= \frac{(b, c, d, e; q)_n}{(aq/b, aq/c, aq/d, aq/e; q)_n}\left(\frac{a^2q^2}{bcde}\right)^n. \end{aligned} \tag{5.13} $$

Using two relations

$$ \begin{aligned} &\frac{(q^{-k}; q)_i}{(deq^{-k}/a; q)_i} = (aq/de)^i \frac{(q; q)_i}{(aq/(de); q)_i}\frac{(q^{i+1}; q)_{k-i}(aq/(de); q)_{k-i}}{(q; q)_{k-i}(aq^{i+1}/(de); q)_{k-i}} \quad (k \ge i), \\ &{}_4\phi_3\left[\begin{matrix} aq/(bc), & d, & e, & q^{-k} \\ & aq/b, & aq/c, & deq^{-k}/a \end{matrix}; q, q\right] \\ &= \sum_{i=0}^{k} \frac{(aq/bc, d, e; q)_i}{(aq/(de), aq/b, aq/c; q)_i}\left(\frac{aq}{de}\right)^i \frac{(q^{i+1}; q)_{k-i}(aq/(de); q)_{k-i}}{(q; q)_{k-i}(aq^{i+1}/(de); q)_{k-i}}, \end{aligned} $$

to reduces (5.13), by exchanging the order of summations and simplifying the resulting sums by the q-Pfaff-Saalschütz formula, to

$$ \begin{aligned} &\sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k}\frac{(aq^n; q)_k}{(aq; q)_k} q^k\, {}_4\phi_3\left[\begin{matrix} aq/(bc), & d, & e, & q^{-k} \\ & aq/b, & aq/c, & deq^{-k}/a \end{matrix}; q, q\right] \\ &\times \frac{(aq, aq/(de); q)_k}{(aq/d, aq/e; q)_k} = \frac{(b, c, d, e; q)_n}{(aq/b, aq/c, aq/d, aq/e; q)_n}\left(\frac{a^2q^2}{bcde}\right)^n. \end{aligned} \tag{5.14} $$

Apply the above Carlitz's inversion to (5.14) and reformulate the resulting identity in terms of standard hypergeometric series as

$$
\begin{aligned}
&{}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq^{n+1} \end{matrix}; q, \frac{a^2q^{n+2}}{bcde}\right] \\
&= \frac{(aq, aq/(de); q)_n}{(aq/d, aq/e; q)_n}\, {}_4\phi_3\left[\begin{matrix} aq/(bc), & d, & e, & q^{-n} \\ & aq/b, & aq/c, & deq^{-n}/a \end{matrix}; q, q\right] \qquad (5.15)
\end{aligned}
$$

Now, define

$$
F(x) = {}_8\phi_7\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & 1/x \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aqx \end{matrix}; q, \frac{a^2q^2x}{bcde}\right];
$$

$$
S(x) = \frac{(aq, aq/(de); q)_\infty}{(aq/d, aq/e; q)_\infty}\frac{(aqx/e, aqx/d; q)_\infty}{(aqx, aqx/(de); q)_\infty}\, {}_4\phi_3\left[\begin{matrix} aq/(bc), & d, & e, & 1/x \\ & aq/b, & aq/c, & de/(ax) \end{matrix}; q, q\right].
$$

It is immediately seen that $F(x)$ is analytic in open set around $x = 0$. So is $S(x)$ if we define particularly

$$
S(0) = \frac{(aq, aq/(de); q)_\infty}{(aq/d, aq/e; q)_\infty}\, {}_3\phi_2\left[\begin{matrix} aq/(bc), & d, & e \\ & aq/b, & aq/c \end{matrix}; q, \frac{a}{de}\right]
$$

and $(de/(ax); q)_k = 0$ at finite points $x = deq^i/a$, which can be guaranteed by the requirement that the ${}_4\phi_3$ series be terminating. Further, Eq.(5.15) means that for each $n \geq 0$, $F(q^n) = S(q^n)$. Then by Theorem 4.1, we have that $F(x) = S(x)$. Substitute x by $1/f$ to get the desired result. ∎

For the limitation of space, we summarize in the following table some well-known summation and transformation formulas which can be shown by the same argument. Formula numbers in these formulas refer to the appendix of Gasper and Rahman's book [22].

TABLE 1. Table of $(f, g) - expansions$

Sums or Transformations	(f,g)-**Expansions**	x_n	b_n
q-Gauss sum II.8	$f = 1, g = x - y$	\	q^n
q-Kummer sum II.9	$f = 1 - xy, g = x - y$	aq^n	q^n
q-analogue of Bailey's ${}_2F_1$ sum II.10	$f = g = x - y$	aq^n	cq^n
q-Dixon sum ${}_4\phi_3$ II.13	$f = 1 - xy, g = x - y$	aq^n	q^n
II.18	$f = (1 - axy)(1 - b\frac{x}{y})$ $g = (x - y)(1 - \frac{b}{axy})$	cq^n	$-q^n$
Bailey's formula ${}_3\varphi_3$ sum II.31	$f = 1 - xy, g = x - y$	q^n	q^n
Bilateral analogue of Dixon's sum II.32		aq^n	q^n
${}_3\varphi_3$ sum II.33		aq^n	q^n
Heine's transformation III.3	$f = 1, g = x - y$	\	cq^n
Jackson's transformation III.4	$f = 1, g = x - y$	\	cq^n
III.9	$f = 1, g = x - y$	\	q^n
III.23	$f = g = x - y$	aq^n	q^n
Singh's transformation III.21	$f = (1 - xy)(1 - qxy), g = x^2 - y^2$	cq^n	q^n
III.35	$f = g = x - y$	aq^n	q^n

5.2. **From nonterminating summations to terminating summations.** The problem stated precisely is that given a nonterminating summation of q-series, as reverse order of the "Ismail's argument", one can always expect a (perhaps new) terminating summation applying the (f, g)-inversion. This method can be sketched as follows: consider that

$$
{}_r\phi_s\left[\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix}; q, z\right]
$$

where q is the base, z is the variable. Clearly, it contains $r+s$ parameters. As it should be, we might choose one of them as a new variable and the former variable z as a parameter such that it can be reformulated as the form of (3.7). If succeed, then we are able to use the (f,g)-expansion formula to derive a (new) terminating summation formula. To illustrate, let take the q-binomial theorem as an example.

Example 5.1. *For $|q|<1$ and a variable $z:|z|<1$,*

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty} \tag{5.16}$$

is equivalent to its finite form (2.17), namely

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k+1}{2}-nk}\begin{bmatrix}n\\k\end{bmatrix}_q(z;q)_k=z^n. \tag{5.17}$$

Proof. Now, we replace the parameter a by a new variable x and take z as a parameter, and then reformulate (5.16) as

$$\sum_{k=0}^{\infty}\frac{z^k}{(q;q)_k}(x;q)_k=\frac{(zx;q)_\infty}{(z;q)_\infty},$$

which turns out to be equivalent to, under the substitution $x\mapsto 1/x, z\mapsto zx$,

$$\sum_{k=0}^{\infty}\frac{(-1)^k z^k}{(q;q)_k}\prod_{i=0}^{k-1}(q^i-x)=\frac{(z;q)_\infty}{(zx;q)_\infty}. \tag{5.18}$$

Letting $x=q^n$ yields a terminating summation formula

$$\sum_{k=0}^{n}\frac{(-1)^k z^k}{(q;q)_k}\prod_{i=0}^{k-1}(q^i-q^n)=(z;q)_n. \tag{5.19}$$

Now, applying the $(1,x-y)$-inversion given in Theorem 1.1 to (5.19) to arrive at

$$\frac{(-1)^n z^n}{(q;q)_n}=\sum_{k=0}^{n}\frac{(z;q)_k}{\prod\limits_{i=0,i\neq k}^{n}(q^i-q^k)},$$

which reduces after simplification to the desired result. Conversely, in the light of the $(1,x-y)$-inversion, (5.17) is equivalent to (5.19), the latter states that (5.18) is valid for $x=q^n, n=0,1,2,\cdots$. Then by the generalized "Ismail's argument", (5.18), i.e., the q-binomial theorem holds. ∎

Clearly, (5.17) is just a q-analogue of the Newton binomial formula

$$\sum_{k=0}^{n}\binom{n}{k}(1-y)^k y^{n-k}=1. \tag{5.20}$$

The next simple example is how to apply our argument to transformations of q-series.

Example 5.2. *Heine's transformation formula of ${}_2\phi_1$ series* (cf.[22, III.1])

$${}_2\phi_1\left[\begin{matrix}a, & b\\ & c\end{matrix};q,z\right]=\frac{(b;q)_\infty(az;q)_\infty}{(c;q)_\infty(z;q)_\infty}{}_2\phi_1\left[\begin{matrix}z, & c/b\\ & az\end{matrix};q,b\right] \tag{5.21}$$

is equivalent to the following finite form of the q-binomial theorem

$$\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q c^k=(c;q)_n. \tag{5.22}$$

Proof. In fact, take the parameter b in (5.21) as a new variable x and define

$$F(x) = \frac{(c;q)_\infty (z;q)_\infty}{(x;q)_\infty (az;q)_\infty}\, {}_2\phi_1\left[\begin{matrix} a, & x \\ & c \end{matrix}; q, z\right].$$

Thus, (5.21) can be rewritten as

$$F(x) = {}_2\phi_1\left[\begin{matrix} z, & c/x \\ & az \end{matrix}; q, x\right]. \tag{5.23}$$

So, it needs only to verify that the r.h.s. of (5.23) is the $(1, x-y)$-expansion formula of $F(x)$ with parameters $b_i = cq^i$ by calculating the n-th order $(1, x-y)$-difference of $F(x)$. The result is

$$\frac{(aq^n z;q)_\infty}{(q^n z;q)_\infty} c^n = \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{j\geq 0} \frac{(a;q)_j (c;q)_{j+k}}{(q;q)_j (c;q)_j} z^j.$$

By employing the q-binomial theorem to expand $(aq^n z;q)_\infty/(q^n z;q)_\infty$ in terms of z^i and then equating the coefficients of z^m on both sides of this identity leads to

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q (cq^m;q)_k = c^n q^{mn}. \tag{5.24}$$

Replace cq^m by c. Then we get its simplified form:

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}-nk} \begin{bmatrix} n \\ k \end{bmatrix}_q (c;q)_k = c^n, \quad \text{i.e., } \mathcal{D}_{q,x}^n \left\{\frac{(c;q)_\infty}{(cx;q)_\infty}\right\}|_{x=1} = c^n, \tag{5.25}$$

which in turn is equivalent to (2.17), i.e., the finite form of the q-binomial theorem. ∎

Performing the argument described in the above examples, we obtain corresponding terminating summations for some well-known summation formulas of basic hypergeometric series (cf.[22, Appendixes I-III]).

Example 5.3. *Heine's q-analogue of the Gauss summation formula* (cf.[22, II.8]) *(previously given as Example 3.1)*

$${}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; q, \frac{c}{ab}\right] = \frac{(c/a, c/b;q)_\infty}{(c, c/(ab);q)_\infty} \tag{5.26}$$

is equivalent to

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \frac{(c/a;q)_k}{(c;q)_k} = q^{\binom{n}{2}} \frac{(a;q)_n (c/a)^n}{(c;q)_n}. \tag{5.27}$$

As a byproduct, let $a \mapsto \infty$ in (5.27). It yields

$$\sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \frac{1}{(c;q)_k} = q^{2\binom{n}{2}} \frac{c^n}{(c;q)_n}. \tag{5.28}$$

Example 5.4. *Jackson's transformation formula of ${}_2\phi_1$ to ${}_2\phi_2$ series* (cf.[22, III.4])

$${}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}; q, z\right] = \frac{(az;q)_\infty}{(z;q)_\infty} {}_2\phi_2\left[\begin{matrix} a, & c/b \\ c, & az \end{matrix}; q, bz\right] \tag{5.29}$$

is equivalent to

$$\sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} \frac{1}{(bq^{n-k};q)_{m+1}} = \begin{bmatrix} m+n \\ m \end{bmatrix}_q \frac{b^n q^{\binom{n}{2}} (q;q)_n}{(b;q)_{m+n+1}}, \quad m \geq 0. \tag{5.30}$$

The details for the proofs of these examples are left to the interested reader. As an interesting application of the expansion formula (3.18), now we can derive two new q-identities from the following indefinite bibasic summation formula of Gasper (cf.[19, Eq.(1.14)])

$$\sum_{k=0}^{m}\frac{(1-ap^kq^k)(1-bp^kq^{-k})}{(1-a)(1-b)}\frac{(a,b;p)_k(x,a/(bx);q)_k}{(q,aq/b;q)_k(ap/x,bpx;p)_k}q^k = \frac{(ap,bp;p)_m(xq,aq/(bx);q)_m}{(q,aq/b;q)_m(ap/x,bpx;p)_m}. \tag{5.31}$$

by the same technique as above.

Theorem 5.5. *For any integers $N, m \geq 0$, it holds*

$$\sum_{K=0}^{N}(-1)^Kq^{\binom{K+1}{2}}\begin{bmatrix}N\\K\end{bmatrix}_q\frac{1-q^{m+1}}{1-q^{m+1+K}}\frac{1-aq^{2K+2m+2}/b}{1-aq^{m+1+K}/b}\frac{(a(pq)^{m+1}q^K,b(p/q)^{m+1}q^{-K};p)_N}{(aq^{2m+2+K}/b;q)_{N+1}} = \frac{(ap^{m+1},bp^{m+1};p)_N}{(aq^{m+1}/b;q)_{N+1}}/\begin{bmatrix}N+m+1\\m+1\end{bmatrix}_q. \tag{5.32}$$

In particular,

$$\mathcal{D}_{q,x}^N\left\{\frac{(a(pq)^{m+1}x;p)_N}{1-xq^{m+1}}\right\}|_{x=1} = \frac{q^{N(m+1)}(ap^{m+1};p)_N}{1-q^{m+1}}/\begin{bmatrix}N+m+1\\m+1\end{bmatrix}_q. \tag{5.33}$$

Proof. Observe that Gasper's indefinite summation formula can be restated shortly as $F(x) = S(x)$ by choosing $f(x,y) = (1-axy)(1-b\frac{x}{y}), g(x,y) = (x-y)(1-\frac{b}{axy})$, in this case, $f(x,y) \in Ker\mathcal{L}_3^{(g)}$, and $b_i = q^i, x_i = p^i$, $|q| < 1$, as well as by defining

$$F(x) = \frac{(ap,bp;p)_m(q/x,aqx/b;q)_m}{(q,aq/b;q)_m(apx,bp/x;p)_m}; \tag{5.34}$$

$$S(x) = \sum_{k=0}^{\infty}G(k)f(x_k,b_k)\frac{\prod_{i=0}^{k-1}g(b_i,x)}{\prod_{i=1}^{k}f(x_i,x)}, \tag{5.35}$$

where the coefficients

$$G(n) = \begin{cases} 0, & \text{for } n \geq m+1; \\ \dfrac{(a,b;p)_na^nq^{n(n+1)/2}}{(1-a)(1-b)(q,aq/b;q)_nb^n}, & \text{for } n \leq m. \end{cases}$$

In the sense of the representation problem of (f, g)-series, $S(x)$ is just the (f, g)-expansion formula of $F(x)$, since $F(x)$ is rational in x, thus analytic around $x = 0$, so is $S(x)$ for the series in the r.h.s. of (5.35) is a finite sum. Thus, by the uniqueness of the (f, g)-expansion formula, namely, Lemma 3.2, we have that for $n \geq m+1$ (the result corresponding to the case $n \leq m$ is trivial),

$$\begin{aligned} G(n) &= \mathbb{D}^{(n)}_{\left((1-axy)(1-b\frac{x}{y}),(x-y)(1-\frac{b}{axy})\right)}\begin{bmatrix}1,q,q^2,\ldots,q^n\\p,p^2,\ldots,p^{n-1}\end{bmatrix}\{F(x)\} \\ &= \frac{1}{(q;q)_n}\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{k+1}{2}-nk}\begin{bmatrix}n\\k\end{bmatrix}_q(1-\frac{b}{a}q^{-2k})\frac{(apq^k,bpq^{-k};p)_{n-1}}{(\frac{b}{a}q^{-k};q^{-1})_{n+1}}F(q^k) = 0. \end{aligned}$$

Putting the values of $F(q^k)$ into it and simplifying the resulted, we obtain that

$$\sum_{k=m+1}^{n}(-1)^{k-m-1}q^{\binom{k}{2}+\binom{m+1}{2}-mk}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k-1\\m\end{bmatrix}_q\frac{1-aq^{2k}/b}{1-aq^k/b}\frac{(ap^{m+1}q^k,bp^{m+1}q^{-k};p)_{n-m-1}}{(aq^{m+k+1}/b;q)_{n-m}} = \frac{(ap^{m+1},bp^{m+1};p)_{n-m-1}}{(aq^{m+1}/b;q)_{n-m}}. \tag{5.36}$$

Next, apply the substitution $n \mapsto N+m+1$ and $k \mapsto K+m+1$ to (5.36), and then use the basic relation

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} k-1 \\ m \end{bmatrix}_q = \begin{bmatrix} n \\ m+1 \end{bmatrix}_q \begin{bmatrix} n-m-1 \\ k-m-1 \end{bmatrix}_q \frac{1-q^{m+1}}{1-q^k}$$

to simplify the resulting identity. The final result is

$$\begin{bmatrix} N+m+1 \\ m+1 \end{bmatrix}_q \sum_{K=0}^{N} (-1)^K q^{\binom{K+1}{2}} \begin{bmatrix} N \\ K \end{bmatrix}_q \frac{1-q^{m+1}}{1-q^{m+1+K}} \frac{1-aq^{2K+2m+2}/b}{1-aq^{K+m+1}/b}$$
$$\frac{(a(pq)^{m+1}q^K, b(p/q)^{m+1}q^{-K}; p)_N}{(aq^{2m+2+K}/b; q)_{N+1}} = \frac{(ap^{m+1}, bp^{m+1}; p)_N}{(aq^{m+1}/b; q)_{N+1}}. \tag{5.37}$$

If we divide both sides of (5.37) by b^{N+1} and then take the limit $b \mapsto 0$ on both sides, then we obtain further

$$\begin{bmatrix} N+m+1 \\ m+1 \end{bmatrix}_q q^{-N(m+1)} \sum_{K=0}^{N} (-1)^K q^{\binom{K+1}{2}-NK} \begin{bmatrix} N \\ K \end{bmatrix}_q$$
$$\frac{1-q^{m+1}}{1-q^{m+1+K}} (a(pq)^{m+1}q^K; p)_N = (ap^{m+1}; p)_N. \tag{5.38}$$

As stated in Proposition 2.2, it can be reformulated in term of $\mathcal{D}_{q,x}$ as the desired form. ∎

6. Conclusions

We hope that the generalized "Ismail's argument" or the representation of analytic functions in terms of (f,g)-series in this article is a new general approach to the basic hypergeometric series. Perhaps, the most intriguing case is that the nth order (f,g)-difference of $F(x) \in \mathcal{H}(\Omega)$

$$\mathbb{D}^{(n)}_{(f,g)} \begin{bmatrix} b_0, b_1, \ldots, b_n \\ x_1, \ldots, x_{n-1} \end{bmatrix} \{F(x)\}$$

can be evaluated in a closed form. If so, it provides an affirmative answer to a problem poised by Marco and Parcet in [35, §5]. On the other hand, the (f,g)-expansion formula of $F(x)$ is in fact a rational approximation to $F(x)$ if $f(x,y)$ and $g(x,y)$ are polynomials of two variables x and y. Thus, it is necessary to study application of this expansion formula as well as the n-th order (f,g)-difference operator in the theory of (numerical) approximation. The same problem remains open to the expansion formula (3.19) in the theory of elliptic hypergeometric series [50]. Besides, we believe our results are also partial solutions to a problem of Gessel and Stanton proposed in §9 of [23]. All these problems will be discussed in our forthcoming papers.

7. Appendix

In this appendix, the proofs of the crucial fact that $\lim_{n\mapsto\infty} S(n,x)/a_n = 0$ invoked in Theorem 4.1 and the fact that Gessel and Stanton's q-analogue (1.3)/(1.4) is equivalent to the $(1-xy, x-y)$-expansion formula with respect to geometric series are displayed in details.

Lemma 7.1. *Preserve all assumptions as in Theorem 4.1 and let $K_{n,k}(x)$ be given by (4.7). Define that*

$$S(n,x) = \sum_{k=1}^{n-1} (-1)^{n-k} A^k p^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_p \frac{(q;q)_{n-k}}{(q;q)_n} \sum_{i=0}^{k} (-1)^i q^{(n-k+1)i+\binom{i}{2}} \begin{bmatrix} k \\ i \end{bmatrix}_q K_{n-k,n-k}(q^i x).$$

Then

$$\lim_{n\mapsto\infty} S(n,x)/a_n = 0 \tag{7.1}$$

for $x = 1, q$.

Proof. We proceed to show the desired result by Tannery's theorem, more precisely, to find $C(n)$ and T_k such that $|t_{n,k}/C(n)| < T_k$, $\sum_{k=0}^{\infty} T_k$ is convergent, and for each fixed k, there holds $\lim_{n\mapsto\infty} t_{n,k}/C(n)$. Obviously, $S(n,x)$ can be reformulated as

$$S(n,x) = \sum_{k=1}^{n-1} t_{n,k};$$
$$t_{n,k} = (-1)^k A^{n-k} p^{\binom{n-k+1}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_p \frac{(q;q)_k}{(q;q)_n} \sum_{i=0}^{n-k} (-1)^i q^{(k+1)i+\binom{i}{2}} \begin{bmatrix} n-k \\ i \end{bmatrix}_q K_{k,k}(q^i x).$$

These required functions can be found by virtue of the following inequalities.

(a) Using the basic relation of the q-binomial coefficients

$$\begin{bmatrix} n+1 \\ i \end{bmatrix}_q = q^i \begin{bmatrix} n \\ i \end{bmatrix}_q + \begin{bmatrix} n \\ i-1 \end{bmatrix}_q$$

and induction on $i \le n$, we obtain at first that

$$\left| \begin{bmatrix} n \\ i \end{bmatrix}_q \right| \le \begin{bmatrix} n \\ i \end{bmatrix}_{|q|}. \tag{7.2}$$

On the other hand, from (2.17) it follows

$$\sum_{i=0}^{n-k} q^{\binom{i}{2}} \begin{bmatrix} n-k \\ i \end{bmatrix}_q x^i = (-x;q)_{n-k}.$$

Combined these two relations. It is easily seen that

$$\sum_{i=0}^{n-k} |q|^{(k+1)i+\binom{i}{2}} \left| \begin{bmatrix} n-k \\ i \end{bmatrix}_q \right| \le \sum_{i=0}^{n-k} |q|^{(k+1)i+\binom{i}{2}} \begin{bmatrix} n-k \\ i \end{bmatrix}_{|q|} = \frac{(-|q|;|q|)_n}{(-|q|;|q|)_k}. \tag{7.3}$$

(b) Note that the real-valued function of a variable x

$$\bar{K}_{k,k}(x) = \sum_{r=0}^{\infty} |a_{r+k}| \begin{bmatrix} r+k \\ r \end{bmatrix}_{|q|} |x|^r$$

is nondecreasing for $|x| \le \min\{1, R\}$, which gives that

$$\bar{K}_{k,k}(q^i x) \le \bar{K}_{k,k}(x) \le \bar{K}_{k,k}(1). \tag{7.4}$$

(**c**) Given $|p| < 1$, the real-valued function of degree two in a variable x (t is as a parameter)

$$y(x) = (x-k)(x-k+1)/2 \ln|p| + (x-k)\ln|A| - t\ln|c_0|x$$

on the interval $x \ge k$ takes its the maximum value M_k at the point $x = k - 1/2 + (t\ln|c_0| - \ln|A|)/\ln|p|$, under the subsidiary condition that $\ln m \le t \le \ln M$, and

$$M_k = (c - 1/2)\ln(|A/c_0|) + (c^2/2 - 1/8)\ln|p| - t\,k\ln|c_0|$$

where $c = (t\ln|c_0| - \ln|A|)/\ln|p|$, since the second derivative $y''(x) = \ln|p| < 0$. It ensures that there must exist a constant $\bar{M}$ independent of k such that

$$|A^{n-k} p^{\binom{n-k+1}{2}}/a_n| = \exp(y(n)) \le \bar{M} R^k. \tag{7.5}$$

So, with the help of Inequalities (7.3)-(7.5), we obtain that

$$
\begin{aligned}
|t_{n,k}| &= \left|(-1)^k A^{n-k} p^{\binom{n-k+1}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_p \frac{(q;q)_k}{(q;q)_n} \sum_{i=0}^{n-k} (-1)^i q^{(k+1)i+\binom{i}{2}} \begin{bmatrix} n-k \\ i \end{bmatrix}_q K_{k,k}(q^i x)\right| \\
&\leq \bar{M} R^k |a_n| \times \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{|p|} \times \frac{|(q;q)_k|}{|(q;q)_n|} \times \bar{K}_{k,k}(x) \times \sum_{i=0}^{n-k} |q|^{(k+1)i+\binom{i}{2}} \left|\begin{bmatrix} n-k \\ i \end{bmatrix}_q\right| \\
&\leq \bar{M} R^k |a_n| \times \frac{(|p|;|p|)_{n-1} |(q;q)_k|}{(|p|;|p|)_{k-1} (|p|;|p|)_\infty |(q;q)_n|} \times \bar{K}_{k,k}(1) \times \frac{(-|q|;|q|)_n}{(-|q|;|q|)_k} = |C(n)| T_k, \qquad (7.6)
\end{aligned}
$$

where, for simplicity, we define

$$
\begin{aligned}
C(n) &= \frac{\bar{M} a_n}{(|p|;|p|)_\infty} \frac{(|p|;|p|)_{n-1}(-|q|;|q|)_n}{|(q;q)_n|}; \\
T_k &= A_k B_k, A_k = \frac{|(q;q)_k||a_k| R^k}{(|p|;|p|)_{k-1}(-|q|;|q|)_k}, B_k = \frac{\bar{K}_{k,k}(1)}{|a_k|}.
\end{aligned}
$$

Note that in the above estimate, we utilize an inequality

$$(|p|;|p|)_{n-k} \geq (|p|;|p|)_\infty \qquad \text{for} \quad |p| < 1.$$

Under the known conditions, we see that $\lim_{k\mapsto\infty} B_k = 1/|(1/R;|q|)_\infty|$ and $\sum_{k\geq 0} A_k$ is convergent. Now, by Cauchy criterion for convergence, it is easily found that $\sum_{k\geq 0} T_k$ is convergent. Now, we are in a position to apply Tannery's theorem to the sum $S(n,x)/C(n)$. The result is

$$
\begin{aligned}
\lim_{n\mapsto\infty} \frac{S(n,x)}{C(n)} &= \frac{|(q;q)_\infty|}{\bar{M}(-|q|;|q|)_\infty} \sum_{k=1}^{\infty} \lim_{n\mapsto\infty} (-1)^k \frac{A^{n-k} p^{\binom{n-k+1}{2}}}{a_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_p \frac{(q;q)_k}{(q;q)_n} \\
&\times \lim_{n\mapsto\infty} \sum_{i=0}^{n-k} (-1)^i q^{(k+1)i+\binom{i}{2}} \begin{bmatrix} n-k \\ i \end{bmatrix}_q K_{k,k}(q^i x) = 0
\end{aligned}
$$

by using the basic relations

$$
\begin{aligned}
&\lim_{n\mapsto\infty} \frac{(-1)^k A^{n-k} p^{\binom{n-k+1}{2}}}{a_n} = 0, \ \lim_{n\mapsto\infty} \frac{K_{n,n}(x)}{a_n} = \frac{1}{(xc_0;q)_\infty}; \\
&\lim_{n\mapsto\infty} \sum_{i=0}^{n-k} (-1)^i q^{(k+1)i+\binom{i}{2}} \begin{bmatrix} n-k \\ i \end{bmatrix}_q K_{k,k}(q^i x) = \frac{(q;q)_\infty}{(q;q)_k} \sum_{r=0}^{\infty} \frac{a_{r+k}}{(q;q)_r} x^r.
\end{aligned}
$$

The last limitation is also a consequence of applying Tannery's theorem. Finally, a simple observation that

$$\lim_{n\mapsto\infty} \frac{S(n,x)}{C(n)} = \frac{|(q;q)_\infty|}{\bar{M}(-|q|;|q|)_\infty} \lim_{n\mapsto\infty} \frac{S(n,x)}{a_n}$$

gives the complete proof of the claimed. ∎

As a previously unknown fact, as mentioned earlier, that the $(1-xy, x-y)-$expansion formula with respect to the parameter sequences $b_i = q^i, x_i = Ap^i$ is actually equivalent to Gessel and Stanton's q-analogue (1.3)/(1.4) of the Lagrange inversion formula for $y = x/(1-x)^{b+1}$(cf.[23, p.180, Theorem 3.7]). Now it can be made clear by reducing the following matrix version of their q-analogue from the $(1-xy, x-y)-$expansion formula in Theorem 4.1.

Lemma 7.2. *Let $B = (B_{n,k})$ and $B^{-1} = (B^{-1}_{n,k})$ be inverses of each other, where*

$$B_{n,k} = \frac{(Ap^k q^k;p)_{n-k}}{(q;q)_{n-k}} q^{-nk}.$$

Then

$$B^{-1}_{n,k} = (-1)^{n-k} q^{\binom{n-k+1}{2}+nk} \frac{(1-Ap^k q^k)(Aq^n p^{n-1};p^{-1})_{n-k-1}}{(q;q)_{n-k}}.$$

Proof. It suffices to, by Theorem 4.1, calculate the nth order $(1-xy, x-y)$-difference of $F(x)$

$$G(n)=\sum_{k=0}^{n}F(q^k)\frac{\prod_{i=1}^{k-1}(1-Ap^iq^k)}{\prod_{i=0}^{k-1}(q^i-q^k)}\frac{\prod_{i=k}^{n-1}(1-Ap^iq^k)}{\prod_{i=k+1}^{n}(q^i-q^k)}. \tag{7.7}$$

Note that

$$\prod_{i=k}^{n-1}(1-Ap^iq^k)=(Ap^kq^k;q)_{n-k},\quad \prod_{i=k+1}^{n}(q^i-q^k)=(-1)^{n-k}q^{k(n-k)}(q;q)_{n-k}.$$

By these notes, (7.7) can be reformulated as

$$(-1)^nG(n)=\sum_{k=0}^{n}B_{n,k}\left\{(-1)^kq^{\binom{k+1}{2}}\frac{(Apq^k;p)_{k-1}}{(q;q)_k}F(q^k)\right\}. \tag{7.8}$$

On the other hand, taking $x=q^n$ in the $(1-xy,x-y)$-expansion formula yields

$$\begin{aligned}F(q^n) &= \sum_{k=0}^{n}(1-Ap^kq^k)\frac{\prod_{i=0}^{k-1}(q^i-q^n)}{\prod_{i=1}^{k}(1-Ap^iq^n)}G(k)\\ &= \frac{\prod_{i=0}^{n-1}(q^i-q^n)}{\prod_{i=1}^{n-1}(1-Ap^iq^n)}\sum_{k=0}^{n}(1-Ap^kq^k)\frac{\prod_{i=k+1}^{n-1}(1-Ap^iq^n)}{\prod_{i=k}^{n-1}(q^i-q^n)}G(k).\end{aligned}$$

A slight simplification gives

$$\frac{\prod_{i=1}^{n-1}(1-Ap^iq^n)}{\prod_{i=0}^{n-1}(q^i-q^n)}F(q^n) = \sum_{k=0}^{n}(1-Ap^kq^k)\frac{\prod_{i=k+1}^{n-1}(1-Ap^iq^n)}{\prod_{i=k}^{n-1}(q^i-q^n)}G(k),$$

which becomes after rewritten in terms of q-shifted factorials

$$q^{-\binom{n}{2}}\frac{(Apq^n;p)_{n-1}}{(q;q)_n}F(q^n) = \sum_{k=0}^{n}(1-Ap^kq^k)q^{-\binom{n}{2}+\binom{k}{2}}\frac{(Aq^np^{n-1};p^{-1})_{n-k-1}}{(q;q)_{n-k}}G(k).$$

Finally, we get

$$\begin{aligned}&\left\{(-1)^nq^{\binom{n+1}{2}}\frac{(Apq^n;p)_{n-1}}{(q;q)_n}F(q^n)\right\}\\ &= \sum_{k=0}^{n}(-1)^{n-k}(1-Ap^kq^k)q^{\binom{n-k+1}{2}+nk}\frac{(Aq^np^{n-1};p^{-1})_{n-k-1}}{(q;q)_{n-k}}\left\{(-1)^kG(k)\right\}.\end{aligned} \tag{7.9}$$

Define that

$$\begin{cases} f_n=(-1)^nG(n);\\ a_n=(-1)^nq^{\binom{n+1}{2}}\frac{(Apq^n;p)_{n-1}}{(q;q)_n}F(q^n),\end{cases}$$

and $X=(f_0,f_1,f_2,\cdots,f_n,\cdots)^T, Y=(a_0,a_1,a_2,\cdots,a_n,\cdots)^T$, the superscript T denotes the transpose of matrix. With these notation, (7.8) and (7.9) can be reformulated respectively as

$$\begin{aligned}f_n &= \sum_{k=0}^{n}B_{n,k}a_k \Leftrightarrow X=BY;\\ a_n &= \sum_{k=0}^{n}(-1)^{n-k}(1-Ap^kq^k)q^{\binom{n-k+1}{2}+nk}\frac{(Aq^np^{n-1};p^{-1})_{n-k-1}}{(q;q)_{n-k}}f_k \Leftrightarrow Y=B^{-1}X.\end{aligned}$$

From this one can read off that

$$B^{-1}_{n,k}=(-1)^{n-k}(1-Ap^kq^k)q^{\binom{n-k+1}{2}+nk}\frac{(Aq^np^{n-1};p^{-1})_{n-k-1}}{(q;q)_{n-k}}.$$

This gives the complete proof of the desired. ∎

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